

Complex Variables
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1 Introduction

This note is intended as a review and reference for the basic theory of complex variables. For further material, and more rigor, Whittaker and Watson is recommended, though there are very many sources available, including a brief review appendix in Matthews and Walker.

2 Complex Numbers

Let z be a **complex number**, which may be written in the forms:

$$z = x + iy \tag{1}$$

$$= re^{i\theta}, \tag{2}$$

where x , y , r , and θ are real numbers. The quantities x and y are referred to as the **real** and **imaginary** parts of z , respectively:

$$x = \Re(z), \tag{3}$$

$$y = \Im(z). \tag{4}$$

The quantity r is referred to as the **modulus** or **absolute value** of z ,

$$r = |z| = \sqrt{x^2 + y^2}, \tag{5}$$

and θ is called the argument, $\theta = \arg(z)$, or the **phase**, or simply the **angle** of z . We have the transformation between these two representations:

$$x = r \cos \theta, \tag{6}$$

$$y = r \sin \theta, \tag{7}$$

and finally also

$$\theta = \tan^{-1}(y/x), \tag{8}$$

with due attention to quadrant. Noticing that $e^{i\theta} = e^{i(\theta+2n\pi)}$, where n is any integer, we say that the **principal value** of $\arg z$ is in the range:

$$-\pi < \arg z \leq \pi. \tag{9}$$

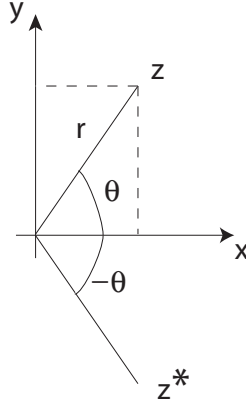


Figure 1: Complex number and its complex conjugate.

The **complex conjugate**, z^* , of z is obtained from z by changing the sign of the imaginary part:

$$z^* = x - iy = r e^{-i\theta}. \quad (10)$$

The product of two complex numbers, z_1 and z_2 , is given by:

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ &= (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1). \end{aligned} \quad (11)$$

Notice that

$$z z^* = x^2 + y^2 = |z|^2. \quad (12)$$

It is also interesting to notice that in the product:

$$z_1 z_2^* = (x_1 x_2 + y_1 y_2) - i(x_1 y_2 - x_2 y_1), \quad (13)$$

the real part looks something like a “scalar product” of two vectors, and the imaginary part resembles a “cross product”.

3 Complex Functions of a Complex Variable

We are interested in (complex-valued) functions of a complex variable z . In particular, we are especially interested in functions which are single-valued, continuous, and possess a derivative in some region.

Defining a suitable derivative requires some care. Start with the definition for real functions of a real number:

$$f'(x) = \frac{df}{dx}(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \quad (14)$$

But in the complex case we have real and imaginary parts to worry about. First, define what we mean by a limit. Let $f(z)$ be a single-valued function defined at all points in a neighborhood of z_0 (except possibly at z_0). Then we say that $f(z) \rightarrow w_0$ as $z \rightarrow z_0$, or $\lim_{z \rightarrow z_0} f(z) = w_0$, if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that (Fig. 2):

$$|f(z) - w_0| < \epsilon \quad \forall z \quad \text{satisfying} \quad 0 < |z - z_0| < \delta. \quad (15)$$

Note that we have not required " $f(z_0)$ " to be defined, in order to define the limit (Fig. 3).

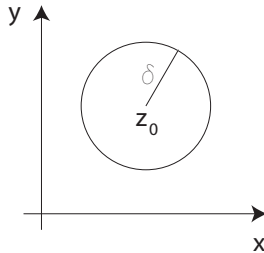


Figure 2: Circle of radius δ about z_0 .

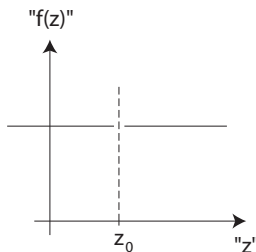


Figure 3: Function not defined at z_0 .

However, in order to define the derivative at z_0 , we require $f(z_0)$ to be defined. If $\lim_{z \rightarrow z_0} f(z) = f(z_0)$, where the limit exists, then we say that $f(z)$ is continuous at z_0 . In general $f(z)$ is complex, and we may write:

$$f(z) = u(x, y) + iv(x, y), \quad (16)$$

where u and v are real. Then $\lim_{z \rightarrow z_0} = f(z_0)$ implies

$$\lim_{x \rightarrow x_0, y \rightarrow y_0} u(x, y) = u(x_0, y_0), \quad (17)$$

$$\lim_{x \rightarrow x_0, y \rightarrow y_0} v(x, y) = v(x_0, y_0), \quad (18)$$

where the path of approach to the limit point must lie within the region of definition. We may thus define continuity to the boundary of a closed region, if the path is within the region.

Now, in our definition of $f'(z)$, we note that there are an infinite number of possible paths along which we can make $\Delta z = \Delta x + i\Delta y \rightarrow 0$.

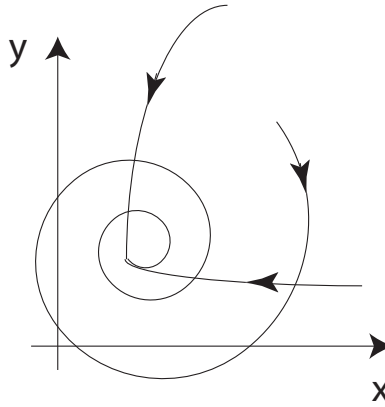


Figure 4: Various paths along which to approach a point.

For our derivative to be well-defined, we demand that the value of $f'(z)$ be independent of the way in which $\Delta z \rightarrow 0$. Thus, if we approach along the path $\Delta x = 0$:

$$\begin{aligned} f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{f(z + i\Delta y) - f(z)}{i\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \left\{ \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \frac{i[v(x, y + \Delta y) - v(x, y)]}{i\Delta y} \right\} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned} \quad (19)$$

If instead we make our approach along the path $\Delta y = 0$, we obtain:

$$\begin{aligned} f'(z) &= \lim_{\Delta x \rightarrow 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned} \quad (20)$$

The two expressions are equal if and only if the real and imaginary parts are separately equal:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (21)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (22)$$

These important conditions are known as the **Cauchy Riemann equations**, or C-R equations, for short. We may state this in the following theorem:

Theorem: If u, v possess first derivatives throughout a neighborhood of z_0 , which are continuous at z_0 , then the Cauchy Riemann equations, if satisfied, guarantee the existence of $\frac{df}{dz}(z_0)$.

Proof: Write:

$$\Delta u = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_{ux} \Delta x + \epsilon_{uy} \Delta y \quad (23)$$

$$\Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \epsilon_{vx} \Delta x + \epsilon_{vy} \Delta y, \quad (24)$$

where the correction terms for non-linearities, ϵ_{ij} , approach zero as $\Delta x, \Delta y \rightarrow 0$.

Using the Cauchy Riemann equations, we obtain:

$$\Delta u = \frac{\partial u}{\partial x} \Delta x - \frac{\partial v}{\partial x} \Delta y + \epsilon_{ux} \Delta x + \epsilon_{uy} \Delta y \quad (25)$$

$$\Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial u}{\partial x} \Delta y + \epsilon_{vx} \Delta x + \epsilon_{vy} \Delta y. \quad (26)$$

Thus,

$$\begin{aligned} \frac{\Delta f}{\Delta z} &= \frac{\Delta u + i\Delta v}{\Delta z} \\ &= \frac{\frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y) + \epsilon_x \Delta x + \epsilon_y \Delta y}{\Delta x + i\Delta y}, \end{aligned} \quad (27)$$

where $\epsilon_x \equiv \epsilon_{ux} + i\epsilon_{vx} \rightarrow 0$, $\epsilon_y \equiv \epsilon_{uy} + i\epsilon_{vy} \rightarrow 0$ as $\Delta x, \Delta y \rightarrow 0$. Furthermore,

$$\left| \frac{\Delta x}{\Delta x + i\Delta y} \right| \leq 1, \quad \left| \frac{\Delta y}{\Delta x + i\Delta y} \right| \leq 1. \quad (28)$$

Therefore,

$$\frac{df}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta u + i\Delta v}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad (29)$$

independent of path. This completes the proof.

We have the following equivalent ways of expressing the derivative:

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}. \quad (30)$$

It is of interest to also consider this discussion in terms of the polar form. In this case, we may consider the $\Delta r = 0$ path:

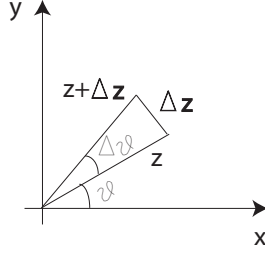


Figure 5: Polar path description.

$$\begin{aligned} f'(z) &= \lim_{\Delta\theta \rightarrow 0} \frac{f(ze^{i\Delta\theta}) - f(z)}{z(e^{i\Delta\theta} - 1)} \\ &= \lim_{\Delta\theta \rightarrow 0} \left\{ \frac{u(r, \theta + \Delta\theta) - u(r, \theta)}{iz\Delta\theta} + \frac{i[v(r, \theta + \Delta\theta) - v(r, \theta)]}{iz\Delta\theta} \right\} \\ &= -\frac{i}{z} \frac{\partial u}{\partial \theta} + \frac{1}{z} \frac{\partial v}{\partial \theta}. \end{aligned} \quad (31)$$

Similarly, for the $\Delta\theta = 0$ path:

$$\begin{aligned} f'(z) &= \lim_{\Delta r \rightarrow 0} \frac{f[(r + \Delta r)e^{i\theta}] - f(z)}{e^{i\theta}\Delta r} \\ &= \lim_{\Delta r \rightarrow 0} \left\{ \frac{u(r + \Delta r, \theta) - u(r, \theta)}{e^{i\theta}\Delta r} + \frac{i[v(r + \Delta r, \theta) - v(r, \theta)]}{e^{i\theta}\Delta r} \right\} \\ &= \frac{1}{e^{i\theta}} \frac{\partial u}{\partial r} + \frac{i}{e^{i\theta}} \frac{\partial v}{\partial r}. \end{aligned} \quad (32)$$

Hence,

$$e^{i\theta} f'(z) = \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = -\frac{i}{r} \frac{\partial u}{\partial \theta} + \frac{1}{r} \frac{\partial v}{\partial \theta}. \quad (33)$$

We have thus obtained the Cauchy-Riemann relations in polar form:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad (34)$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}. \quad (35)$$

A function $f(z)$ of complex variable z is called **analytic** at the point z_0 if it is single-valued and possesses a derivative at every point in a neighborhood of z_0 . Otherwise, z_0 is a **singular point** of $f(z)$. If $f(z)$ is analytic at every point in a simply connected open region (“domain”) D , then it is referred to as analytic throughout D . Other terms that are often used for this (with some variation of meaning) are **regular** and **holomorphic**. Sometimes the term “analytic” is not required to be single-valued, that is, single-valuedness in a domain D means that, after following any closed path in D , the function $f(z)$ returns to its initial value. If $f(z)$ is analytic for all finite z , then $f(z)$ is an **entire function**.

Examples:

- $f(z) = z^3$ is an entire function.
- $f(z) = 1/z^2$ is analytic everywhere except at $z = 0$, where it is not defined. We note that for this function,

$$u = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad v = -\frac{2xy}{(x^2 + y^2)^2}. \quad (36)$$

- $f(z) = z^{3/2}$ is analytic everywhere except at $z = 0$. Let’s look at why this is the case in some detail. We may write

$$z^{3/2} = r^{3/2}e^{3i\theta/2} = r^{3/2}(\cos 3\theta/2 + i \sin 3\theta/2). \quad (37)$$

Hence,

$$\frac{\partial u}{\partial r} = \frac{3}{2}r^{1/2} \cos 3\theta/2 \quad (38)$$

$$\frac{\partial v}{\partial r} = \frac{3}{2}r^{1/2} \sin 3\theta/2 \quad (39)$$

$$\frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{3}{2}r^{1/2} \sin 3\theta/2 \quad (40)$$

$$\frac{1}{r} \frac{\partial v}{\partial \theta} = \frac{3}{2}r^{1/2} \cos 3\theta/2. \quad (41)$$

Comparison with Eqns. 34 and 35 shows that the C-R conditions are satisfied everywhere. Now consider a path containing the origin as an interior point (see Fig. 6). We’ll start at $z = \epsilon e^{i0}$, with ϵ real. Table 1 shows the values of $f(z)$ as we traverse the path once around the origin. We see that $f(z) = z^{3/2}$ is multi-valued in any neighborhood of the origin, and hence is not analytic at $z = 0$.

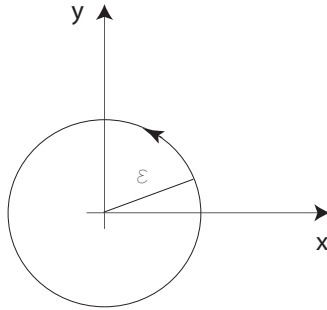


Figure 6: Circular path around origin, radius ϵ .

Table 1: Evaluation of the function $z^{3/2}$ at various points on a circle.

θ	z	$f(z) = z^{3/2}$
0	ϵ	$\epsilon^{3/2}$
$\pi/2$	$i\epsilon$	$\epsilon^{3/2}e^{i3\pi/4}$
π	$-\epsilon$	$\epsilon^{3/2}e^{i3\pi/2} = -i\epsilon^{3/2}$
$3\pi/2$	$-i\epsilon$	$\epsilon^{3/2}e^{i9\pi/4}$
2π	ϵ	$\epsilon^{3/2}e^{i3\pi} = -\epsilon^{3/2}$

4 Riemann Surfaces

Let us continue to think about the interesting $f(z) = z^{3/2}$ example. Note that if $\theta = 4\pi$ and $r = \epsilon$, then $z^{3/2} = \epsilon^{3/2}e^{i6\pi} = \epsilon^{3/2}$, so we come back to the $\theta = 0$ value after two circuits. Thus $z^{3/2}$ is a double-valued function. We may visualize this behavior via the use of **Riemann surfaces**, or **sheets**. For $z^{3/2}$, we have two sheets (Fig. 7).

The point $z = 0$ is called a **branch point**. Since there are only a finite number of **branches** (2) for this function, the origin is called an **algebraic branch point**.

For another example, the function $f(z) = z^{1/4}$ will have four branches, see Table 2 and Fig. 8.

Now consider the function $f(z) = \ln z$, defined by $e^{f(z)} = z$:

$$f(z) = \ln z = \ln r + i\theta. \quad (42)$$

This function has an infinite number of branches. In this case, the point $z = 0$ is called a **logarithmic branch point**.

We note a couple of things about branches:

- There may be many branch points for a function.

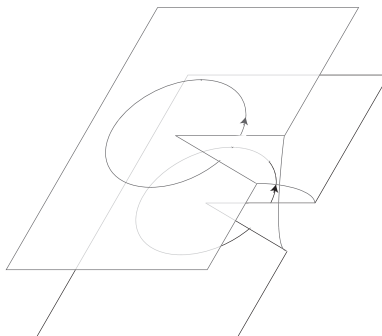


Figure 7: Two Riemann sheets for the double-valued function $z^{3/2}$. The lower sheet is for $0 \leq \theta < 2\pi$, $4\pi \leq \theta < 6\pi$, etc., and the top sheet is for $2\pi \leq \theta < 4\pi$, etc. The branch cuts are indicated by the cuts in the planes.

Table 2: The function $f(z) = z^{1/4}$, evaluated at multiples of 2π .

θ	$e^{i\theta/4}$
0	1
2π	$e^{i\pi/2} = i$
4π	$e^{i\pi} = -1$
6π	$e^{i3\pi/2} = -i$
8π	$e^{i2\pi} = 1$

- There are many ways to make branch cuts, but they can only terminate at a branch point, they cannot intersect themselves, and they must have the same form on all sheets.

For a slightly more complicated example illustrating these ideas, consider the function:

$$f(z) = \sqrt{1 - z^2} = (1 - z)^{1/2}(1 + z)^{1/2}. \quad (43)$$

This function has singularities (branch points) at $z = \pm 1$. There are two sheets, and various possible ways of choosing the branch cuts, as illustrated in Fig. 9.

There is a choice in how to take branch cuts – one makes cuts that are convenient to the problem at hand (for example, when we integrate along a path, we arrange it so that the path does not cross a cut). Branch cuts are used in effect to make multi-valued functions “single-valued” – if you don’t cross a branch, you stay on the same sheet.

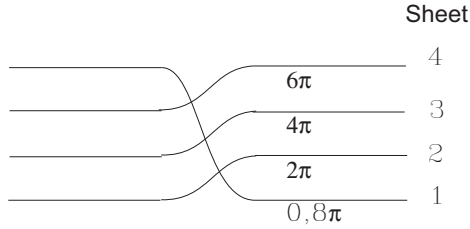


Figure 8: Four Riemann sheets for the quadruple-valued function $z^{1/4}$. The view is edge-on, with the branch cut at the transitions among the sheets.

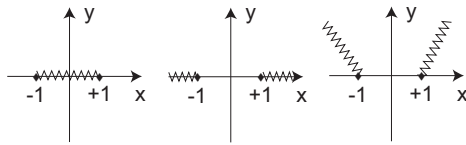


Figure 9: Some possible choices of branch cuts for the function $\sqrt{1-z^2}$.

5 Integration of Complex Functions

As with the derivative, we must face the problem of forming an integral that makes sense in some correspondence with the integral for real functions. For real functions, the indefinite integral may be “defined” as the inverse of differentiation (*i.e.*, as the limit of a sum, rather than the limit of a difference). For a complex function, such an indefinite integral may not always exist.

Consider $f(z) = z^* = x - iy$. Suppose

$$F(z) = \int f(z) dz = U + iV, \quad (44)$$

and (if integration is inverse of differentiation)

$$\frac{dF}{dz} = f(z) = x - iy. \quad (45)$$

If the derivative exists, we must be able to use the Cauchy-Riemann equations, hence,

$$\frac{dF}{dz} = \frac{\partial U}{\partial x} - i \frac{\partial U}{\partial y} = x - iy. \quad (46)$$

Thus,

$$\frac{\partial U}{\partial x} = x, \quad \frac{\partial U}{\partial y} = y, \quad (47)$$

and

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 2. \quad (48)$$

Let us see what the Cauchy-Riemann equations imply for this quantity:

$$\frac{\partial}{\partial x} \left[\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \right] \quad (49)$$

$$\frac{\partial}{\partial y} \left[\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \right]. \quad (50)$$

Therefore:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad (51)$$

which may be recognized as **Laplace's Equation** in two dimensions. Thus, $F(z)$ cannot be an analytic function; it does not possess a derivative, and there exists no function with derivative $x - iy$. This suggests that we should restrict consideration to functions which are analytic in the region of interest.

Referring to Fig. 10, let us consider the definite integral:

$$\int_{\alpha}^{\beta} f(z) dz. \quad (52)$$

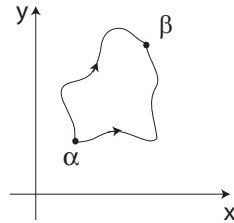


Figure 10: Possible paths of integration from α to β .

There are an infinite number of possible paths to integrate along. In general, we must specify the path, *e.g.*,

$$\int_C^{\beta} f(z) dz. \quad (53)$$

To define this integral, first divide path C into n intervals by points $z_0 = \alpha, z_1, z_2, \dots, z_n = \beta$, as in Fig. 11. Let $\Delta_j z \equiv z_j - z_{j-1}$, and let z'_j be

a point on C between z_{j-1} and z_j . Then we define the line integral along C as:

$$\int_C^\beta f(z)dz = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(z'_j) \Delta_j z, \quad (54)$$

where we require the intervals to satisfy:

$$\lim_{n \rightarrow \infty} \max_{j=1}^n |\Delta_j z| = 0, \quad (55)$$

and the limit must exist, of course. Note that this definition is compatible with the usual definition for real variables.

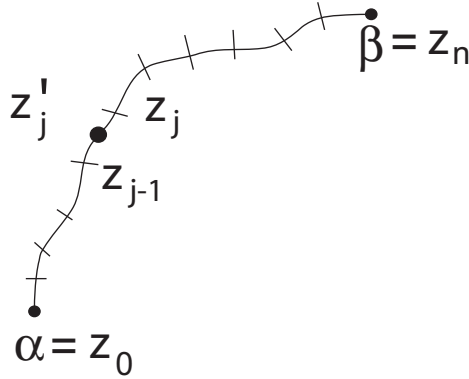


Figure 11: Dividing a path into intervals to obtain an approximate integral.

We list some immediate consequences of our definition:

1. Considering $\Delta_j z \rightarrow -\Delta_j z$, we have the path-reversed integral:

$$\int_C^\alpha f(z)dz = -\int_C^\beta f(z)dz. \quad (56)$$

2. If k is any complex constant, then

$$\int_C^\beta k f(z)dz = k \int_C^\beta f(z)dz. \quad (57)$$

3. If the integrals of f and g separately exist, then the integral of their sum exists, and:

$$\int_C^\beta [f(z) + g(z)] dz = \int_C^\beta f(z)dz + \int_C^\beta g(z)dz. \quad (58)$$

4. If γ is a point on C (between α and β), then

$$\int_C^\alpha^\gamma f(z)dz + \int_C^\gamma^\beta f(z)dz = \int_C^\alpha^\beta f(z)dz. \quad (59)$$

Toward proving this, note that we can always arrange our subintervals such that γ is a dividing point.

5. If $M = \max_C^\beta |f(z)|$ (including the endpoints) then:

$$\begin{aligned} \left| \int_C^\alpha^\beta f(z)dz \right| &= \left| \lim_{n \rightarrow \infty} \sum_{j=1}^n f(z'_j) \Delta_j z \right| \\ &\leq \lim_{n \rightarrow \infty} \sum_{j=1}^n |f(z'_j) \Delta_j z| \quad (\text{follows from triangle inequality}) \\ &\leq M \int_C^\alpha^\beta |dz| = ML_C, \end{aligned} \quad (60)$$

where L_C is the length of the integration path (in the usual Euclidean sense).

6 Cauchy's Theorem

If a function $f(z)$ is analytic at all points on and inside a contour C , then

$$\int_C f(z)dz = 0. \quad (61)$$

Note that by “contour”, we mean a simple closed curve. We could also use the notation \oint to stress this. Our assertion is known as **Cauchy's Theorem**.

Let us prove the theorem: Assume $f(z)$ is analytic as stated. Write $f(z) = u(x, y) + iv(x, y)$. Then:

$$\begin{aligned} \int_C f(z) dz &= \int_C (u + iv)(dx + idy) \\ &= \int_C (udx - vdy) + i \int_C (udy + vdx) \end{aligned} \quad (62)$$

Let S stand for the region enclosed by contour C . Green's theorem states that, for functions α and β :

$$\int_C \alpha dx + \beta dy = \int_S \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) dx dy. \quad (63)$$

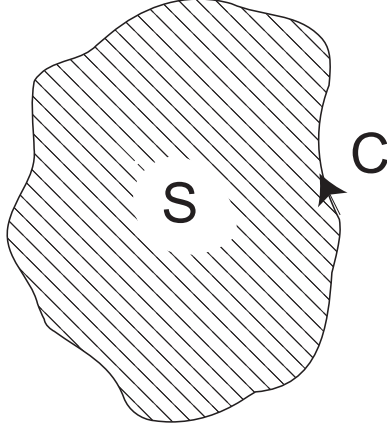


Figure 12: Contour and surface of integration in Green's theorem.

Therefore,

$$\begin{aligned} \int_C f(z) dz &= - \int_S \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx dy + i \int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy \\ &= 0, \quad \text{by the Cauchy-Riemann relations.} \end{aligned} \quad (64)$$

Cauchy's theorem tells us that the integral of an analytic function is path-independent in a domain of analyticity:

$$\begin{aligned} \int_{\alpha}^{\beta} f(z) dz &= \int_{\alpha}^{\beta} f(z) dz \\ &= \int_{\alpha}^{\beta} f(z) dz, \end{aligned} \quad (65)$$

where the latter equality is without ambiguity, due to Cauchy's theorem.

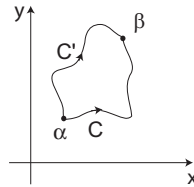


Figure 13: Equivalent paths of integration in a region of analyticity.

Note that the way we have stated Cauchy's theorem, it holds for functions which have singularities, provided our contours do not "encircle" the singularities:

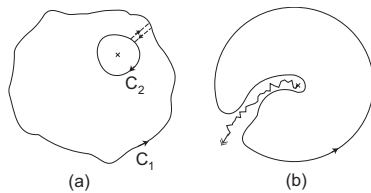


Figure 14: (a) $\int_{C_1+C_2} f(z) dz = 0$, where C_2 encircles a singularity, but the “contour” $C_1 + C_2$ does not. Integrals along the portions joining C_1 and C_2 cancel out. (b) A branch cut may be chosen for convenience, so that the contour does not cross it.

7 Indefinite Integral of an Analytic Function

Let $f(z)$ be analytic in simply connected domain D . Then

$$F(z) = \int_{z_0}^z f(z') dz' \quad (66)$$

depends only on z and z_0 (and not the path), as long as the path is entirely in D .

What is $F'(z) = \frac{dF}{dz}(z)$ (for $z \in D$)?

$$\begin{aligned} \frac{F(z + \Delta z) - F(z)}{\Delta z} &= \frac{1}{\Delta z} \left[\int_{z_0}^{z+\Delta z} f(z') dz' - \int_{z_0}^z f(z') dz' \right] \\ &= \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z') dz' \\ &= f(z) + \frac{1}{\Delta z} \int_z^{z+\Delta z} [f(z') - f(z)] dz'. \end{aligned} \quad (67)$$

The last integral is path independent in D , so chose for path the straight line segment joining z and $z + \Delta z$ (noting that, for Δz small enough, such a path must exist). Thus, by the continuity of $f(z)$, given an $\epsilon > 0$, we can always find $|\Delta z|$ small enough such that $|f(z') - f(z)| < \epsilon$ for any z' on the path. Thus,

$$\left| \int_z^{z+\Delta z} [f(z') - f(z)] dz' \right| < \epsilon |\Delta z|. \quad (68)$$

Given any $\epsilon > 0$ then, we can find a $\Delta z > 0$ such that

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| < \epsilon. \quad (69)$$

Therefore,

$$F'(z) = \frac{d}{dz} \int_{z_0}^z f(z') dz' = f(z). \quad (70)$$

The indefinite integral of an analytic function is an analytic function.

It is important, when performing integrations, to be careful about singularities and regions of non-analyticity. For example, consider the integral $\int_{-1}^1 \frac{1}{z} dz$. We might try an integration path along a semi-circle in the positive y plane – $1/z$ is analytic there.

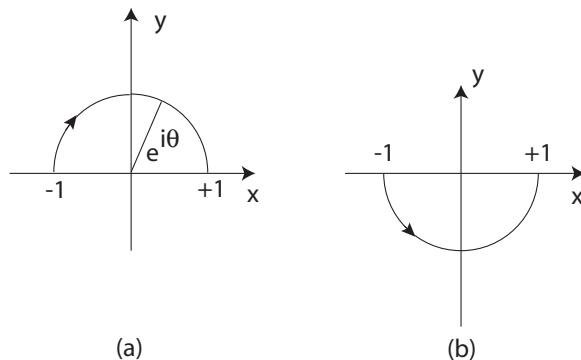


Figure 15: Two possible semi-circular paths from -1 to $+1$.

We let $z = e^{i\theta}$, and hence $dz = ie^{i\theta} d\theta$. Alternatively, we could choose to integrate along a semicircle in the negative y plane – $1/z$ is analytic there as well. The two choices yield:

$$I_+ = i \int_{\pi}^0 e^{-i\theta} e^{i\theta} d\theta = -i\pi \quad (71)$$

$$I_- = i \int_{\pi}^{2\pi} e^{-i\theta} e^{i\theta} d\theta = i\pi. \quad (72)$$

The two answers are different! The path-dependence is a result of the fact that we have chosen paths which lie in *different* simply-connected domains of analyticity. There is a branch cut from the origin, a singular point. Note that, while $1/z$ is not multi-valued, its integral ($\ln z$) is.

8 Cauchy Integral Formula

Suppose $f(z)$ is analytic everywhere in some domain D . Consider the integral:

$$\int_C \frac{f(z)}{z - z_0} dz, \quad (73)$$

where C is contained in D , and z_0 is interior to C . Thus, $\frac{f(z)}{z-z_0}$ is analytic everywhere on and inside C , except at the point $z = z_0$. The integral is unchanged if we deform the contour to the circle C_0 with center at z_0 :

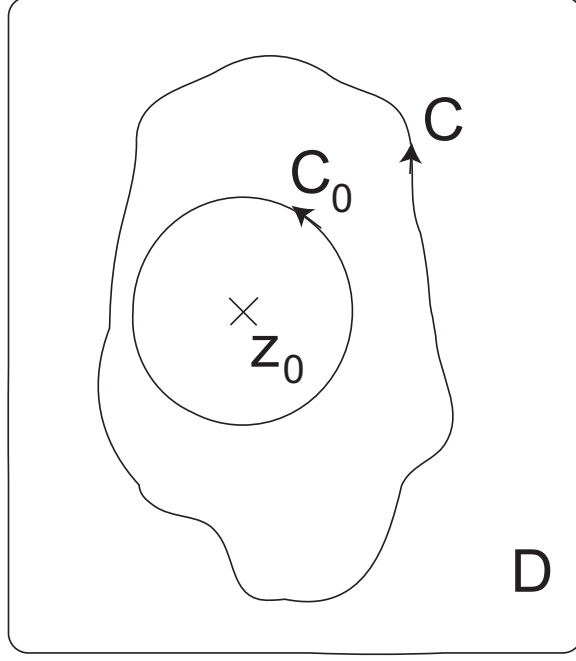


Figure 16: Domain D , contour C and deformed contour C_0 about point z_0 .

$$\begin{aligned} \int_C \frac{f(z)}{z-z_0} dz &= \int_{C_0} \frac{f(z)}{z-z_0} dz \\ &= \int_{C_0} \frac{f(z_0)}{z-z_0} dz + \int_{C_0} \frac{f(z) - f(z_0)}{z-z_0} dz. \end{aligned} \quad (74)$$

Consider the second of the two integrals in the above expression:

$$\left| \int_{C_0} \frac{f(z) - f(z_0)}{z-z_0} dz \right| \leq \int_{C_0} \frac{|f(z) - f(z_0)|}{|z-z_0|} |dz|. \quad (75)$$

Since $f(z)$ is analytic at z_0 , it must be continuous there. Hence, given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. We pick an ϵ , and let $\delta = |z - z_0|$, *i.e.*, we pick a circle of small enough radius such that $|f(z) - f(z_0)| < \epsilon$ on the circle. Remember that the value of the

integral does not depend on the radius of the circle. Thus,

$$\left| \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{\epsilon}{\delta} \int_{C_\delta} |dz| \leq 2\pi\epsilon. \quad (76)$$

The integral is smaller than any positive number, *i.e.*, is equal to zero. Therefore,

$$\begin{aligned} \int_{C_0} \frac{f(z)}{z - z_0} dz &= f(z_0) \int_{C_0} \frac{dz}{z - z_0} \\ &= f(z_0) \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} \quad (\text{letting } z - z_0 = re^{i\theta}) \\ &= 2\pi i f(z_0). \end{aligned} \quad (77)$$

We have derived **Cauchy's Integral Formula**: For any function $f(z)$ which is analytic on and inside the contour C ,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz. \quad (78)$$

Note that the Cauchy integral formula tells us that if we know the value of a function everywhere along a closed contour, then we know its value at every point inside the contour, provided the function is analytic on and inside the contour.

8.1 Cauchy Integral Formula and Derivatives of an Analytic Function

Start with Cauchy's integral formula (assuming $f(z)$ appropriately analytic), and take derivatives:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z} dz' \quad (79)$$

$$\frac{df}{dz} = \frac{1}{2\pi i} \int_C \frac{f(z')}{(z' - z)^2} dz' \quad (80)$$

$$\frac{d^2 f}{dz^2} = \frac{2}{2\pi i} \int_C \frac{f(z')}{(z' - z)^3} dz' \quad (81)$$

$$\dots$$

$$\frac{d^n f}{dz^n} = \frac{n!}{2\pi i} \int_C \frac{f(z')}{(z' - z)^{n+1}} dz'. \quad (82)$$

Is this procedure justified? If so, then we have evidently shown that the derivative of an analytic function is analytic, at least at all points *inside* C .

If $f(z)$ is analytic, we know its derivative exists:

$$\begin{aligned}
f'(z) &= \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{2\pi i h} \left[\int_C \frac{f(z') dz'}{z' - z - h} - \int_C \frac{f(z') dz'}{z' - z} \right] \\
&= \frac{1}{2\pi i} \lim_{h \rightarrow 0} \left[\int_C \frac{f(z') dz'}{(z' - z - h)(z' - z)} \right] \tag{83}
\end{aligned}$$

Adding and subtracting $f(z')/(z' - z)^2$ to the integrand, we obtain:

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(z') dz'}{(z' - z)^2} + \lim_{h \rightarrow 0} \frac{h}{2\pi i} \int_C \frac{f(z') dz'}{(z' - z - h)(z' - z)^2}. \tag{84}$$

By assumption, $f(z')$ is continuous on C , hence it is bounded. Likewise, $(z' - z)^{-2}$ is bounded on C . Furthermore, take $h < \min_C \frac{1}{2}|z' - z|$, guaranteeing that $|z' - z - h| > 0$. Therefore,

$$\left| \frac{f(z')}{(z' - z)^2(z' - z - h)} \right| \leq K < \infty, \tag{85}$$

i.e., the integrand is bounded for z' on C by some finite number K . Then,

$$\left| \lim_{h \rightarrow 0} \frac{h}{2\pi i} \int_C \frac{f(z') dz'}{(z' - z - h)(z' - z)^2} \right| \leq \frac{K}{2\pi} \lim_{h \rightarrow 0} |h| L_C = 0, \tag{86}$$

where L_C is the length of contour C . Hence,

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{(z' - z)^2} dz', \tag{87}$$

as desired.

Then we may similarly consider:

$$\begin{aligned}
\lim_{h \rightarrow 0} \frac{f'(z+h) - f'(z)}{h} &= \lim_{h \rightarrow 0} \frac{1}{2\pi i h} \int_C f(z') dz' \left[\frac{1}{(z' - z - h)^2} - \frac{1}{(z' - z)^2} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{2\pi i} \int_C f(z') dz' \frac{2(z' - z - h/2)}{(z' - z)^2(z' - z - h)^2} \\
&= \frac{2}{2\pi i} \int_C \frac{f(z')}{(z' - z)^3} dz' + \lim_{h \rightarrow 0} h A_h, \tag{88}
\end{aligned}$$

where A_h is a bounded function of z when $h < \frac{1}{2}|z' - z|$. Hence, f'' exists, and

$$f'' = \frac{2}{2\pi i} \int_C \frac{f(z')}{(z' - z)^3} dz'. \tag{89}$$

The same argument may be continued indefinitely, since the integral representation has $f(z)$, which we know is continuous, hence bounded. Thus, we have established the result for the n^{th} derivative:

$$f^{(n)} = \frac{n!}{2\pi i} \int_C \frac{f(z')}{(z' - z)^{n+1}} dz', \quad (90)$$

as hoped.

8.2 Mean Value Theorem from the Cauchy Integral Formula

If $f(z)$ is analytic on and within contour C , we know that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z') dz'}{z' - z}. \quad (91)$$

Consider contour C that is a circle of radius r with center at z_0 :

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_C \frac{f(z') dz'}{z' - z_0} \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z') ire^{i\theta} d\theta}{re^{i\theta}} = \frac{1}{2\pi r} \int_0^{2\pi} f(z') r d\theta \\ &= \frac{1}{2\pi r} \int_C f(z') ds, \end{aligned} \quad (92)$$

where ds is an element of circular arc. Thus, $f(z_0)$ is given by the average value of $f(z)$ on a circle centered at z_0 (entirely within the domain of analyticity).

9 Taylor Series

Let $f(z)$ be analytic in domain D with $z_0 \in D$, and circle $C \subset D$ centered at z_0 [hence, $f(z)$ is analytic within and on C]. Let $z = z_0 + h$ be interior to C . Then, use Cauchy's integral:

$$\begin{aligned} f(z) &= f(z_0 + h) = \frac{1}{2\pi i} \int_C \frac{f(z') dz'}{z' - z_0 - h} \\ &= \frac{1}{2\pi i} \int_C f(z') dz' \left[\frac{1}{z' - z_0} + \frac{h}{(z' - z_0)^2} + \cdots \right. \\ &\quad \left. + \frac{h^n}{(z' - z_0)^{n+1}} + \frac{h^{n+1}}{(z' - z_0)^{n+1}(z' - z_0 - h)} \right], \end{aligned} \quad (93)$$

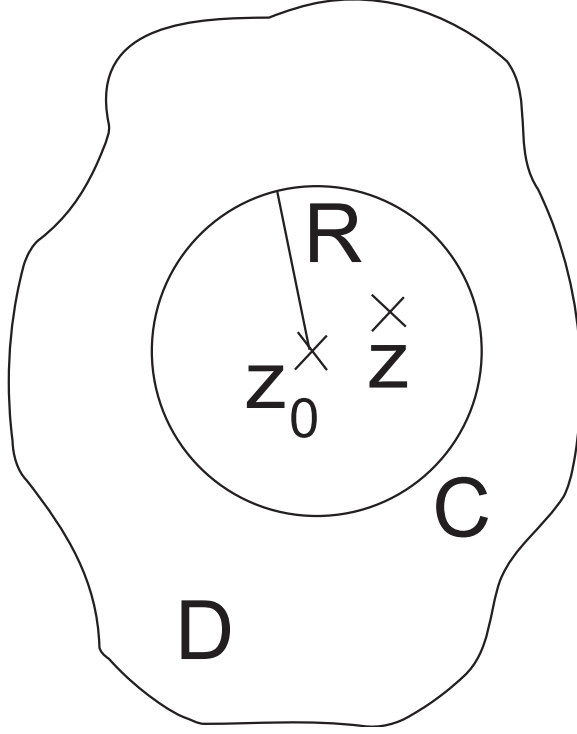


Figure 17: Illustration for Taylor series discussion.

where we have used

$$\frac{1}{(z' - z_0 - h)} = \frac{1}{(z' - z_0)} + \frac{h}{(z' - z_0)(z' - z_0 - h)}. \quad (94)$$

We also know that:

$$f^{(n)} = \frac{n!}{2\pi i} \int_C \frac{f(z')}{(z' - z)^{n+1}} dz', \quad (95)$$

Comparing, we have:

$$\begin{aligned} f(z) &= f(z_0) + hf^{(1)}(z_0) + \frac{h^2}{2}f^{(2)}(z_0) + \cdots + \frac{h^n}{n!}f^{(n)}(z_0) \\ &+ \frac{h^{n+1}}{2\pi i} \int_C \frac{f(z') dz'}{(z' - z_0)^{n+1}(z' - z_0 - h)}. \end{aligned} \quad (96)$$

Thus, we have

$$f(z) = \sum_{k=0}^n (z - z_0)^k \frac{f^{(k)}(z_0)}{k!} + R_n, \quad (97)$$

where

$$R_n = \frac{(z - z_0)^{n+1}}{2\pi i} \int_C \frac{f(z') dz'}{(z' - z_0)^{n+1}(z' - z)}. \quad (98)$$

We see that term by term this is the same form as the Taylor series expansion for a real function of a real variable.

Let us investigate the remainder term, R_n . In particular, how big is it? We first notice that $f(z')$ and $1/|z' - z|$ are continuous, hence bounded, on C :

$$\left| \frac{f(z')}{z' - z} \right| \leq M, \quad z' \in C, z \text{ inside } C. \quad (99)$$

Let R be the radius of C . Then:

$$\begin{aligned} |R_n| &= \frac{1}{2\pi} \left| (z - z_0)^{n+1} \int_C \frac{f(z') dz'}{(z' - z_0)^{n+1}(z' - z)} \right| \\ &\leq \frac{M}{2\pi} |z - z_0|^{n+1} \frac{1}{R^{n+1}} 2\pi R \\ &\leq MR \left| \frac{z - z_0}{R} \right|^{n+1}. \end{aligned} \quad (100)$$

Since $\left| \frac{z - z_0}{R} \right| < 1$, we can approximate $f(z)$ to any desired accuracy with our finite Taylor series expansion.

10 Bolzano-Weierstrass Theorem

We wish to consider infinite series next, which means we must concern ourselves with issues of convergence. Let us begin with sequences. Given any sequence of complex numbers, $z_1, z_2, \dots \equiv \{z_n\}$, we say that the sequence $\{z_n\}$ tends to the limit L as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} z_n = L, \quad (101)$$

if, for every $\epsilon > 0$, there exists N such that $|z_{N+k} - L| < \epsilon$ for all positive integers k . If $\{z_n\}$ is such that for any real number G , we can find N so that $|z_{N+k}| > G$ for all positive integers k , then we say that $|z_n|$ **tends to ∞ as $n \rightarrow \infty$** .

Finally, if a sequence does not tend to a unique limit, and does not tend to plus or minus infinity, then the sequence is said to **oscillate**.

Definition: A **limit point** of a set \mathcal{S} is a point such that there are an unlimited number of elements of \mathcal{S} which are arbitrarily close to the limit point.

For example, 1 is a limit point for the sequence $1 + 1/n$ (even though 1 is not an element of the sequence). For another example, 1 is a limit point for the sequence $1, 2, 1, 2, 1, 2, 1, 2, \dots$

Theorem: (Bolzano-Weierstrass) If $\{x_n\}$ is an infinite sequence of real numbers, and there exists a, b such that $a \leq x_n \leq b$ for all n (where a and b are independent of n), then $\{x_n\}$ has at least one limit point.

Proof: Let G be a real number such that $G > |a|$, $G > |b|$. Then, $G > |x_n|$ for all n . Consider the interval $I_0 = (-G, G)$. Cut it in half (say, to $(-G, 0)$ and $[0, G)$): At least one subinterval must contain an infinite number of members of the sequence $\{x_n\}$. Call the rightmost such interval I_1 . Now cut I_1 in half. Again, at least one subinterval must contain an infinite number of members of the sequence $\{x_n\}$. Call the rightmost such interval I_2 . We may continue this interval subdivision indefinitely, making our interval as small as we please. In the nested set of intervals I_1, I_2, I_3, \dots there exists a point L which belongs to all the intervals of the nest. Choose k sufficiently large such that the length of I_k is less than any given $\epsilon > 0$. Then if $\{x'_n\}_k$ is the infinite set of members of $\{x_n\}$ which lies in I_k , we have that $|x'_n - L| < \epsilon$ for all members of $\{x'_n\}_k$. Hence L is a limit point of the sequence.

11 Cauchy's Condition for the Existence of a Limit, or, Cauchy's Principle of Convergence

Theorem: A sequence of complex numbers z_1, z_2, \dots has a limiting value if and only if, given any $\epsilon > 0$ there is an N such that $|z_{N+k} - z_N| < \epsilon$ for all positive integers k .

This convergence condition is referred to as Cauchy's condition. Note the distinction between this theorem and the definition of the limiting value. To apply this test, one does not need to know, *a priori*, what the limit is.

Proof: Necessity: We suppose a limit, L , exists. Then, given any $\epsilon > 0$, there exists an N such that $|z_N - L| < \epsilon/2$, and $|z_{N+k} - L| < \epsilon/2$ for all positive integers k . By the triangle inequality:

$$|z_{N+k} - z_N| \leq |z_{N+k} - L| + |z_N - L| < \epsilon. \quad (102)$$

Sufficiency: We suppose that given an $\epsilon > 0$ there exists an N such that $|z_{N+k} - z_N| < \epsilon$ for all positive integers k . But the hypotenuse of a triangle is longer than either other leg, and hence:

$$\epsilon > |z_{N+k} - z_N| \geq |x_{N+k} - x_N| \quad (103)$$

$$\geq |y_{N+k} - y_N|. \quad (104)$$

Thus, we may consider a real sequence $\{x_n\}$ which satisfies the Cauchy condition. Consider $\epsilon = 1$, and pick an $N = M$ such that:

$$|x_{M+k} - x_M| < 1 \quad \forall k = 1, 2, 3, \dots \quad (105)$$

Let a_1, b_1 be the least and greatest values, respectively, of the finite sequence x_1, x_2, \dots, x_M . Let $a = a_1 - 1$ and $b = b_1 + 1$. Then $a < x_n < b$ for all n . By the Bolzano-Weierstrass theorem, $\{x_n\}$ has at least one limit point, G .

Now we must demonstrate that there is only one limit point: Suppose there are at least two, G and H . Then, given $\epsilon > 0$, there exists an n such that $|x_{n+p} - x_n| < \epsilon$, by hypothesis, and there exists positive integers q and r such that $|G - x_{n+q}| < \epsilon$ and $|H - x_{n+r}| < \epsilon$, since G and H are limit points. Thus,

$$\begin{aligned} |G - H| &= |G - x_{n+q} + x_{n+q} - x_n + x_n - x_{n+r} + x_{n+r} - H| \\ &\leq |G - x_{n+q}| + |x_{n+q} - x_n| + |x_{n+r} - x_n| + |H - x_{n+r}| \\ &< 4\epsilon. \end{aligned} \quad (106)$$

Hence $G=H$, and there is only one limit point. Thus, given $\delta > 0$, there are at most a finite number of terms of the sequence outside the interval $(G - \delta, G + \delta)$, so G is the limit of $\{x_n\}$.

Similarly, the imaginary part sequence has a limit, hence $\{z_n\}$ has a limit [noting that if $\lim_{n \rightarrow \infty} z_n = L$, and $\lim_{n \rightarrow \infty} z'_n = L'$, then $\lim_{n \rightarrow \infty} (z_n + z'_n) = L + L'$].

12 Infinite Series

Given a sequence $\{u_n\}$, we can construct a sequence:

$$S_0 = u_0 \quad (107)$$

$$S_1 = u_0 + u_1 \quad (108)$$

\vdots

$$S_n = \sum_{k=0}^n u_k. \quad (109)$$

These are the “partial sums” of the **infinite series**:

$$S = \sum_{k=0}^{\infty} u_k. \quad (110)$$

The infinite series is said to **converge** if, given $\epsilon > 0$ there exists S and n_0 such that:

$$|S - S_n| < \epsilon, \quad \forall n > n_0. \quad (111)$$

If the series $\sum_{n=0}^{\infty} u_n$ converges: It is said to be **absolutely** convergent if $\sum_{n=0}^{\infty} |u_n|$ converges; otherwise it is **conditionally** convergent. Note that an absolutely convergent series may be rearranged at will, with identical results, but this doesn't hold for a conditionally convergent series.

We give some tests for convergence, leaving the proofs to the reader:

1. **Cauchy Integral test** for convergence: If $f(x)$ is a positive, real, decreasing function of x for real $x \geq 1$, then the series $S = \sum_{n=1}^{\infty} f(n)$ converges or diverges, depending on whether the integral

$$\lim_{n \rightarrow \infty} \int_1^n f(x) dx \quad (112)$$

converges or diverges.

2. **Comparison test** for absolute convergence: $S = \sum_{n=0}^{\infty} u_n$ is absolutely convergent if

$$|u_n| < c|v_n|, \quad \forall n > N, \quad (113)$$

c is independent of n , and $\sum_{n=0}^{\infty} v_n$ is known to be absolutely convergent.

3. **d'Alembert's ratio test** for absolute convergence: $\sum_{n=0}^{\infty} u_n$ converges absolutely if

$$\overline{\lim}_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1, \quad (114)$$

where $\overline{\lim}$ is the “limit superior”, or least upper bound of all convergent subsequences of $\{u_n\}$. The sum diverges if

$$\underline{\lim}_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1. \quad (115)$$

4. **Raabe's test** for absolute convergence: If

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = 1, \quad (116)$$

and

$$\overline{\lim}_{n \rightarrow \infty} n \left(\left| \frac{u_{n+1}}{u_n} \right| - 1 \right) < -1, \quad (117)$$

then $\sum_{n=0}^{\infty} u_n$ converges absolutely.

5. **Cauchy's test** for absolute convergence: If

$$\overline{\lim}_{n \rightarrow \infty} |u_n|^{1/n} < 1, \quad (118)$$

then $\sum_{n=0}^{\infty} u_n$ converges absolutely.

13 Series of Functions

If the terms of an infinite series are functions of complex variable z , then the series may converge or not, depending on the value of z . We are interested in the **region of convergence** of such a series. We are also interested in continuity, integrability, and differentiability of such a series (especially of analytic functions, including power series).

If $S(z) = \sum_{n=0}^{\infty} u_n(z)$ and $S_N(z) = \sum_{n=0}^N u_n(z)$, then $S(z)$ is said to be **uniformly convergent** over the set of points $\{z | z \in R\} = R$ if, given any $\epsilon > 0$, there exists an N such that:

$$|S(z) - S_{N+k}(z)| < \epsilon, \quad \forall k = 0, 1, 2, \dots, \text{ and } \forall z \in R. \quad (119)$$

Note that the condition of uniform convergence is in a sense stronger than simple convergence – $S(z)$ may converge for all $z \in R$, without being uniformly convergent. As an example, consider $f(z) = 1/(1 - z)$, for $R = \{z : |z| < 1\}$.

A necessary and sufficient condition for uniform convergence is “Cauchy's principle for uniform convergence: Given $S(z) = \sum_{n=0}^{\infty} u_n(z)$ which converges for all $z \in R$, where R is a closed region, and any $\epsilon > 0$, then $S(z)$ converges uniformly in R if there exists an N such that

$$|S_N(z) - S_{N+k}(z)| < \epsilon, \quad \forall k = 0, 1, 2, \dots, \text{ and } \forall z \in R. \quad (120)$$

It is left to the reader to prove this, using techniques similar to methods already encountered.

Another, sufficient, test for uniform convergence is the “Weierstrass M test”: If $|u_n(z)| \leq M_n$, where M_n is a positive real number, independent of $z \in R$, and if $\sum_{n=0}^{\infty} M_n$ converges, then $S(z)$ is uniformly convergent on $z \in R$.

Let us consider the following example: Suppose we have the real series

$$S(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots \quad (121)$$

$$= \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}. \quad (122)$$

We see that $S(x)$ converges absolutely for all real x , since:

$$S_N(x) = \sum_{n=0}^N \frac{x^2}{(1+x^2)^n} \quad (123)$$

$$= \begin{cases} 0 & \text{if } x = 0, \\ 1 + x^2 - \frac{1}{(1+x^2)^N} & \text{if } x \neq 0. \end{cases} \quad (124)$$

Thus, $S(x)$ converges absolutely for all possible real x values.

But does this series converge uniformly? We suspect trouble because of the peculiar behavior at $x = 0$:

$$S(x) = \begin{cases} 0 & \text{at } x = 0, \\ 1 + x^2 & \text{for } x \neq 0. \end{cases} \quad (125)$$

That is, $S(x)$ is discontinuous at $x = 0$. For uniform convergence, we must have the case that, given any $\epsilon > 0$ there exists an N , independent of x , such that

$$|S_N(x) - S_{N+k}(x)| < \epsilon, \quad \forall k = 0, 1, 2, \dots, \text{ and } \forall x. \quad (126)$$

Assume $x > 0$. Then:

$$|S_N(x) - S_{N+k}(x)| = \frac{1}{(1+x^2)^{N+k}}. \quad (127)$$

Let's choose $\epsilon = 1/2$. Notice that for any fixed N , and any chosen k , we can always pick $x > 0$ small enough so that

$$\frac{1}{(1+x^2)^{N+k}} > \frac{1}{2} = \epsilon. \quad (128)$$

Hence the convergence is not uniform near $x = 0$.

The following theorem addresses the question of continuity:

Theorem: If $S(z) = \sum_{n=0}^{\infty} u_n(z)$ is a uniformly convergent series of continuous functions $u_n(z)$ for all $z \in R$, where R is a closed region, then $S(z)$ is a continuous function of z , for all $z \in R$.

Proof: Write $S(z) = S_N(z) + R_N(z)$, where $R_N(z) = \sum_{k=1}^{\infty} u_{n+k}(z)$. $S_N(z)$ is a finite sum of continuous functions and hence is continuous throughout R . By uniform convergence, given any $\epsilon > 0$, we can find N such that $R_N(z) < \frac{1}{3}\epsilon$, for all $z \in R$. Furthermore, since $S_N(z)$ is continuous for all $z \in R$, there exists a $\rho > 0$ such that $|S_N(z + \delta) - S_N(z)| < \frac{1}{3}\epsilon$ whenever $|\delta| < \rho$ and $z + \delta \in R$. Therefore:

$$\begin{aligned} |S(z + \delta) - S(z)| &= |S_N(z + \delta) - S_N(z) + R_N(z + \delta) - R_N(z)| \\ &\leq |S_N(z + \delta) - S_N(z)| + |R_N(z + \delta)| + |R_N(z)| \\ &< \epsilon. \end{aligned} \quad (129)$$

Hence, $S(z)$ is continuous for all $z \in R$.

We may also be concerned with the question of multiplication of series. If two series are absolutely convergent, then the series formed of product terms is absolutely convergent independent of order, and the product series is equal to the product of the individual series.

Next, let us consider the integration of a series:

Theorem: Let $S(z) = \sum_{n=0}^{\infty} u_n(z)$ be a uniformly convergent series of continuous functions in a domain D . Then, if $C \subset D$, where C is a finite path in D , we have:

$$\int_C S(z)dz = \sum_{n=0}^{\infty} \int_C u_n(z)dz. \quad (130)$$

The order of integration and summation may be interchanged for a series of continuous functions in its domain of uniform convergence.

Proof: Write $S(z) = \sum_{k=0}^n u_k(z) + R_n(z)$, where $R_n(z) = \sum_{k=1}^{\infty} u_{n+k}(z)$. Then

$$\int_C S(z)dz = \sum_{k=0}^n \int_C u_k(z)dz + \int_C R_n(z)dz. \quad (131)$$

Since $S(z)$ is uniformly convergent, for any given $\epsilon > 0$, there exists an N such that $R_n(z) < \epsilon$ for all $n \geq N$ and all $z \in D$. Now, if $L_C = \int_C |dz| < \infty$ is the "path length", then

$$\left| \int_C R_n(z)dz \right| < \epsilon L_C, \quad \forall n \geq N. \quad (132)$$

Hence,

$$\left| \int_C S(z)dz - \sum_{k=0}^n \int_C u_k(z)dz \right| < \epsilon L_C, \quad \forall n \geq N, \quad (133)$$

which can be made arbitrarily small.

Next, we investigate the differentiation of a series.

Theorem: If $S(z) = \sum_{n=0}^{\infty} u_n(z)$ is a series of functions which are analytic on and inside a contour C , and if $S(z)$ converges uniformly on C , then $S(z)$ is analytic everywhere inside C , with derivative:

$$\frac{dS}{dz}(z) = \sum_{n=0}^{\infty} \frac{du_n}{dz}(z). \quad (134)$$

That is, The order of differentiation and summation may be reversed.

Proof: Let z_0 be a point inside C .

$$\begin{aligned}
\frac{1}{2\pi i} \int_C \frac{S(z)dz}{z - z_0} &= \frac{1}{2\pi i} \int_C \left[\sum_{n=0}^{\infty} u_n(z) \right] \frac{dz}{z - z_0} \\
&= \frac{1}{2\pi i} \left[\int_C \sum_{k=0}^n \frac{u_k(z)}{z - z_0} dz + \int_C \frac{R_n(z)}{z - z_0} dz \right] \\
&= \sum_{k=0}^n u_k(z_0) + \frac{1}{2\pi i} \int_C \frac{R_n(z)}{z - z_0} dz. \tag{135}
\end{aligned}$$

The series converges uniformly on C , so given any $\epsilon > 0$ there exists an N such that $|R_k(z)| < \epsilon$ for all $k \geq n$ and for all $z \in C$. Thus,

$$\left| \int_C \frac{R_n(z)}{z - z_0} dz \right| < \epsilon \left| \int_C \frac{dz}{z - z_0} \right| < 2\pi\epsilon. \tag{136}$$

Therefore, the series converges, and we have:

$$\frac{1}{2\pi i} \int_C \frac{S(z)}{z - z_0} dz = \sum_{n=0}^{\infty} u_n(z_0) \equiv S(z_0), \tag{137}$$

where we take the latter as the definition of $S(z_0)$ interior to C .

Thus, $S(z) = \sum_{n=0}^{\infty} u_n(z)$ is defined on and inside C . To prove analyticity inside C , we show that the derivative exists:

$$\begin{aligned}
S'(z_0) &= \lim_{h \rightarrow 0} \frac{S(z_0 + h) - S(z_0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{2\pi i} \frac{1}{h} \int_C \left[\frac{S(z)}{z - z_0 - h} - \frac{S(z)}{z - z_0} \right] dz \\
&= \lim_{h \rightarrow 0} \frac{1}{2\pi i} \int_C \frac{S(z)}{(z - z_0 - h)(z - z_0)} dz \\
&= \frac{1}{2\pi i} \left[\sum_{k=0}^n \int_C \frac{u_k(z)}{(z - z_0)^2} dz + \int_C \frac{R_n(z)}{(z - z_0)^2} dz \right]. \tag{138}
\end{aligned}$$

The first term is the form of the derivative of an analytic function we saw earlier. The second term can be made arbitrarily small by taking n large enough, by the uniform convergence of S on C . Hence,

$$S'(z) = \sum_{n=0}^{\infty} \frac{du_n}{dz}(z). \tag{139}$$

Let us now turn to the special case of power series, of which the Taylor series is an important example.

Theorem: If $S(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $z = z_1$, then it is absolutely convergent for all $|z| < |z_1|$.

Proof: Since the series converges for $z = z_1$, $a_n z_1^n$ must be bounded: $|a_n z_1^n| < M$ for all n . Pick any z such that $|z| < |z_1|$. Let $r = |z|/|z_1| < 1$. Then

$$|a_n z^n| = |a_n z_1^n| \left| \frac{z}{z_1} \right|^n < M r^n, \quad (140)$$

and

$$\sum_{n=0}^{\infty} |a_n z^n| < \sum_{n=0}^{\infty} M r^n = M \sum_{n=0}^{\infty} r^n. \quad (141)$$

Since $r < 1$, this is convergent, hence $S(z)$ is absolutely convergent for all $|z| < |z_1|$.

A similar argument can be used to show that, if $S(z) = \sum_{n=0}^{\infty} a_n z^n$ diverges for $z = z_1$, then it diverges for all $|z| > |z_1|$. Thus, the region of convergence of a power series is a circle: Inside the circle there is absolute convergence, and outside there is divergence. On the circle, we cannot say in general. For example,

$$S(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \begin{cases} \text{diverges for } |z| > 1 \\ \text{absolutely converges for } |z| < 1 \\ \text{converges for } z = -1 \\ \text{diverges for } z = +1. \end{cases} \quad (142)$$

We state and leave it for the reader to prove the following:

Theorem: A power series is uniformly convergent in any closed region inside the circle of convergence.

We have the following uniqueness theorem for power series:

Theorem: If $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges for all points inside the circle $|z - z_0| = r_0$, then the series is the Taylor series for $S(z)$ (about z_0).

Proof: The proof consists in differentiating k times, and showing that $a_n = S^{(n)}(z_0)/n!$.

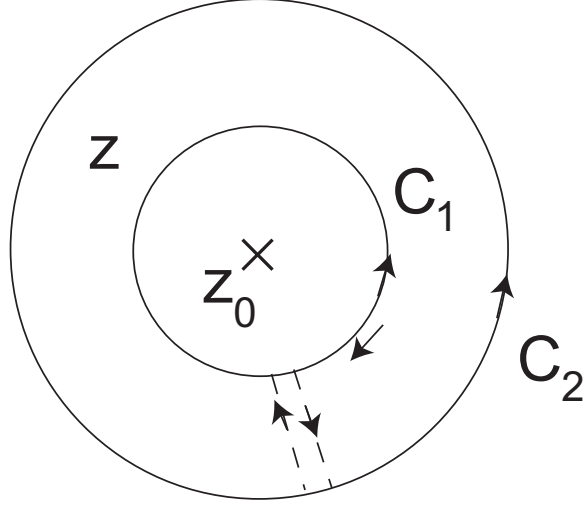


Figure 18: Illustration for Laurent series discussion.

14 Laurent Series

We now introduce a generalization of the Taylor series, the **Laurent series**. Consider a function $f(z)$ which is analytic in a region containing two concentric circles (but not necessarily in the interior of the smaller circle).

“Contour” $C_2 - C_1$ (Fig. 18 represents a closed path in a “simply-connected” domain, so we can use the Cauchy Integral Formula (for z in the annulus):

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(z')}{z' - z} dz' - \frac{1}{2\pi i} \int_{C_1} \frac{f(z')}{z' - z} dz'. \quad (143)$$

Now,

$$\frac{1}{z' - z} = \frac{1}{z' - z_0} \cdot \frac{1}{1 - (z - z_0)/(z' - z_0)}. \quad (144)$$

For z' on C_2 , z in the annulus, and with z_0 the center of the circles, $|(z - z_0)/(z' - z_0)| < 1$. We may thus write, for z' on C_2 :

$$\frac{1}{z' - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z' - z_0)^{n+1}}. \quad (145)$$

Similarly, for z' on C_1 :

$$-\frac{1}{z' - z} = \sum_{n=0}^{\infty} \frac{(z' - z_0)^n}{(z - z_0)^{n+1}}. \quad (146)$$

Putting this back into 143:

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{(z - z_0)^n}{2\pi i} \int_{C_2} \frac{f(z')}{(z' - z_0)^{n+1}} dz' - \frac{1}{2\pi i} \frac{1}{(z - z_0)^{n+1}} \int_{C_1} (z' - z_0)^n f(z') dz' \right]. \quad (147)$$

Thus, we can write:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}, \quad (148)$$

where:

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z')}{(z' - z_0)^{n+1}} dz', \quad n = 0, 1, 2, \dots \quad (149)$$

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z')}{(z' - z_0)^{-n+1}} dz', \quad n = 1, 2, \dots \quad (150)$$

Or, we may combine the series:

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n, \quad (151)$$

where,

$$A_n = \frac{1}{2\pi i} \int_C \frac{f(z')}{(z' - z_0)^{n+1}} dz', \quad (152)$$

where C is any contour which makes one counter-clockwise passage around z_0 , and lies in the region bounded by C_1 and C_2 . This is called the **Laurent series**.

If we express $f(z) = \phi(z) + \psi(z)$, where

$$\phi(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n, \quad (153)$$

$$\psi(z) = \sum_{n=1}^{\infty} A_{-n} (z - z_0)^{-n}, \quad (154)$$

then $\psi(z)$ is called the **principal part** of $f(z)$. Note that $\phi(z)$ converges uniformly in any closed region interior to the outer edge of the annulus. Hence, $f(z) = \phi(z) + \psi(z)$ converges uniformly in any closed region within the annulus.

If $z = z_0$ is a singularity of $f(z)$, and there exists a neighborhood of z_0 which contains no other singularity, then z_0 is called an **isolated singularity** of $f(z)$. For example, $z = 1$ is an isolated singularity of $f(z) = 1/(z - 1)$. If

all the coefficients of the principal part vanish, then an isolated singularity z_0 is called a **removable singularity**. For example, the origin is a removable singularity of $f(z) = \sin z/z$. The singularity in this case may be “removed” by defining

$$f(0) \equiv \lim_{z \rightarrow 0} \frac{\sin z}{z} = 1. \quad (155)$$

If the principal part terminates after a finite number of terms, say

$$A_{-m} \neq 0, \quad (156)$$

$$A_{-(m+k)} = 0, \quad \forall k = 1, 2, 3, \dots, \quad (157)$$

then $f(z)$ is said to have a **pole of order m** at z_0 . For example, $f(z) = 1/(z - z_0)^2$ has a pole of order 2 at z_0 .

If the principal part has an infinite number of non-vanishing coefficients, then z_0 is called an **essential singularity** of $f(z)$. An essential singularity need not be isolated. For example, $z = 0$ is an essential singularity of $f(z) = 1/\sin(1/z)$. It is also the limit point of a sequence of poles, and hence is not an isolated singularity. On the other hand, $z = 0$ is an isolated essential singularity of $f(z) = e^{1/z}$.

If the Laurent series is not known, the order of a pole may be determined by examining limits. Consider the limits:

$$\lim_{z \rightarrow z_0} (z - z_0)^n f(z), \quad n = 1, 2, 3, \dots \quad (158)$$

The lowest n for which the limit exists is the order of the pole at z_0 .

15 Residues

Consider the integral:

$$I_n = \int_C (z - z_0)^n dz, \quad (159)$$

where C is a closed contour surrounding $z = z_0$ and n is an integer. Since $(z - z_0)^n$ is analytic, except possibly at z_0 , we may deform the contour into a circle centered at z_0 without affecting I_n . Then we may write $z - z_0 = Re^{i\theta}$, and hence

$$I_n = iR^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta. \quad (160)$$

$$= \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}. \quad (161)$$

Now suppose we have a function, $f(z)$, which is analytic in a region except at the point z_0 in the region. Then we can make the Laurent expansion about z_0 :

$$f(z) = \sum_{n=-\infty}^{\infty} A_n(z - z_0)^n. \quad (162)$$

Take a contour C around z_0 :

$$\frac{1}{2\pi i} \int_C f(z) dz = \frac{1}{2\pi i} \int_C \sum_{n=-\infty}^{\infty} A_n(z - z_0)^n dz \quad (163)$$

$$= A_{-1}. \quad (164)$$

Thus the coefficient of $1/(z - z_0)$ in the Laurent series is given by $A_{-1} = \frac{1}{2\pi i} \int_C f(z) dz$. This coefficient is called the **residue** of $f(z)$ at z_0 . Notice that the residue is zero if $f(z)$ is analytic at z_0 , or if the coefficient A_{-1} is zero (even if z_0 is a pole or isolated essential singularity).

We now come to the important and useful **residue theorem**. Consider contour C in a region where $f(z)$ is analytic except at isolated singularities (poles or essential singularities).

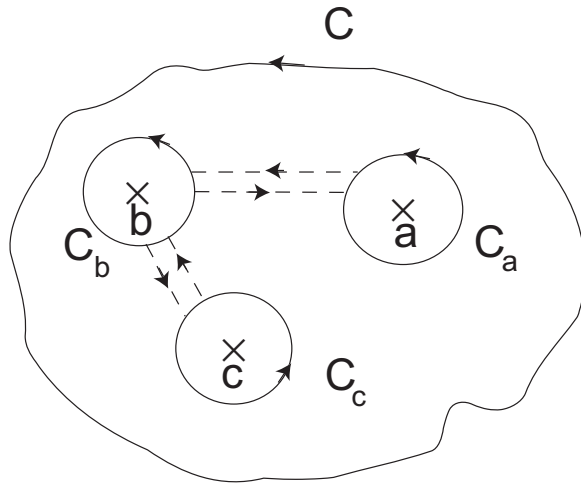


Figure 19: Contours to illustrate residue theorem. Singularities are at a , b , and c .

We want to determine $\int_C f(z) dz$. We can write this in terms of the sum of the integrals around each singularity:

$$\int_C f(z) dz = \int_{C_a} f(z) dz + \int_{C_b} f(z) dz + \dots$$

$$\begin{aligned}
&= 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) \\
&= 2\pi i \sum_{\text{singularities}} R,
\end{aligned} \tag{165}$$

where $a_{-1}, b_{-1}, c_{-1}, \dots$ are the residues at a, b, c, \dots , respectively, and $\sum R$ is the sum of the residues of $f(z)$ interior to the contour C .

The computation of the residues is thus often an important part of evaluating integrals. At a simple pole, the Laurent series is

$$f(z) = \frac{A_{-1}}{z - z_0} + \sum_{n=0}^{\infty} A_n(z - z_0)^n, \tag{166}$$

and hence

$$A_{-1} = \lim_{z \rightarrow z_0} [(z - z_0)f(z)]. \tag{167}$$

For a pole of order m ,

$$f(z) = \frac{A_{-m}}{(z - z_0)^m} + \frac{A_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{A_{-1}}{z - z_0} + \sum_{n=0}^{\infty} A_n(z - z_0)^n. \tag{168}$$

If we multiply both sides by $(z - z_0)^m$, we have:

$$(z - z_0)^m f(z) = A_{-m} + A_{-m+1}(z - z_0) + \cdots + A_{-1}(z - z_0)^{m-1} + \sum_{n=0}^{\infty} A_n(z - z_0)^{n+m}. \tag{169}$$

Now differentiate $m - 1$ times and evaluate at $z = z_0$:

$$\frac{d^{m-1} [(z - z_0)^m f(z)]}{dz^{m-1}} \Big|_{z=z_0} = (m - 1)! A_{-1}, \tag{170}$$

and hence,

$$A_{-1} = \frac{1}{(m - 1)!} \frac{d^{m-1} [(z - z_0)^m f(z)]}{dz^{m-1}} \Big|_{z=z_0}. \tag{171}$$

But sometimes it is easier to just carry out the expansion sufficiently to find A_{-1} directly.

16 Cauchy Principal Value Integral

Suppose $f(z)$ has a simple pole at $z = z_0 = x_0 + i0$ on the real axis. We may define an integral along the real axis through this pole according to:

$$P \int_{\alpha}^{\beta} f(z) dz \equiv \lim_{\epsilon \rightarrow 0} \left[\int_{\alpha}^{x_0 - \epsilon} f(x) dx + \int_{x_0 + \epsilon}^{\beta} f(x) dx \right], \tag{172}$$

where $\alpha < x_0 < \beta$. This is known as the **Cauchy Principal Value Integral**.