Ph129 PS 6 solutions

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Problem 25

(a)

For $\hat{p}$ to be Hermitian, we have to have $<\hat{p}\phi|\psi>=<\phi|\hat{p}\psi>$. Expanding the left side and integrating by parts we see that:

$$<\hat{p}\phi|\psi>=-i\psi(x)\phi^*(x)|_a^b+<\phi|\hat{p}\psi>, \tag{1}$$

so $\hat{p}$ is Hermitian as long as $\psi(x)\in D_{\hat{p}}$ and $\phi(x)\in D_{\hat{p}^\dagger}$ satisfy:

$$\psi(x)\phi^*(x)|_a^b=0. \tag{2}$$

(b)

For the momentum operator to be self-adjoint, we need to satisfy the above hermiticity condition and $D_{\hat{p}}=D_{\hat{p}^\dagger}$ by definition of a self-adjoint operator. Thus we need to have

$$\psi(b)=\psi(a). \tag{3}$$

(c)

First we note that if the square of the operator is bounded, then the operator itself must be bounded. Since the square of the momentum operator is proportional to the kinetic energy, we see that the operator is not bounded. Consider for example an infinite square well or a harmonic oscillator. The energy in each of these systems is not bounded and thus the kinetic energy is not bounded. Therefore, the momentum operator is not bounded.

Problem 26

(a)

$f(t)$ is periodic with period T. We are going to find the Fourier series expansion for $f(t)$ restricted to the interval $[-T/2, T/2]$. Let ω be the Fourier transformed variable for t. The Fourier series is expanded by the exponentials $e^{-i\omega_n t}$. These satisfy the periodic boundary condition

$$e^{-i\omega_n t} = e^{-i\omega_n (t+T)} \; . \tag{4}$$

So, $\omega_n = 2\pi n/T$. The Fourier Series of $f(t)$ is given by

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} s_n e^{-2\pi i n t/T} \\ s_n &= \frac{1}{T}\int_{-T/2}^{T/2} dt\; e^{2\pi i n t/T} f(t) \end{aligned} \tag{5}$$

Note that

$$f(t) = \begin{cases} \frac{2}{T} + \frac{4}{T^2}t & \text{if } t \in [-T/2, 0] \\ \frac{2}{T} - \frac{4}{T^2}t & \text{if } t \in [0, T/2] \end{cases} \; . \tag{6}$$

The integration to get s_n is straightforward and the result is

$$s_n = \frac{4}{\pi^2 n^2 T}\sin^2\frac{\pi n}{2} = \begin{cases} 0 & \text{if n even} \\ \frac{4}{\pi^2 n^2 T} & \text{if n odd} \end{cases} \tag{7}$$

(b)

Let's treat $d(t)$ just as an ordinary function.

$$\begin{aligned} \hat{d}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt e^{i\omega t} d(t) \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\; e^{i\omega t}\delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\omega n T} \end{aligned} \tag{8}$$

$\hat{d}(\omega)$ has also an expression in terms of delta functions as shown in (18).

(c)

$f(t)$ is a infinite sum of some translations of $h(t)$:

$$f(t) = \sum_{n=-\infty}^{\infty} h(t - nT) \ . \tag{9}$$

This can be put into the form of convolution:

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} dt' \sum_{n=-\infty}^{\infty} \delta(t' - nT) h(t - t') \\ &= \int_{-\infty}^{\infty} dt' \; d(t') h(t - t') \end{aligned} \tag{10}$$

(d)

The Fourier transform $\hat{h}(\omega)$ of h(t) is given by

$$\begin{aligned} \hat{h}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} h(t) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{0} dt \; e^{i\omega t} \left(\frac{2}{T} + \frac{4}{T^2} t \right) + \frac{1}{\sqrt{2\pi}} \int_{0}^{T/2} dt \; e^{i\omega t} \left(\frac{2}{T} - \frac{4}{T^2} t \right) \\ &= \frac{1}{\sqrt{2\pi}} \frac{16}{\omega^2 T^2} \sin^2 \frac{\omega T}{4} \ . \end{aligned} \tag{11}$$

The convolution theorem is

$$\hat{f}(\omega) = \sqrt{2\pi} \hat{f}_1(\omega) \hat{f}_2(\omega) \qquad \text{for } f(t) = \int_{-\infty}^{\infty} f_1(t') f_2(t - t') dt' \ . \tag{12}$$

The Fourier transform of the convolution (10) is then

$$\begin{aligned} \hat{f}(\omega) &= \sqrt{2\pi} \hat{d}(\omega) \hat{h}(\omega) \\ &= \frac{1}{\sqrt{2\pi}} \frac{16}{\omega^2 T^2} \sum_{n=-\infty}^{\infty} e^{i\omega n T} \sin^2 \frac{\omega T}{4} \ . \end{aligned} \tag{13}$$

Note that, from (7), the n th Fourier component corresponding to $\omega = 2\pi i n / T$ is

$$s_\omega = \frac{16}{\omega^2 T^3} \sin^2 \frac{\omega T}{4} \ . \tag{14}$$

This is the same expression as $\hat{f}(\omega)$ except that it does not have the factor $\hat{d}(\omega)T = \frac{T}{\sqrt{2\pi}} \sum e^{i\omega nT}$. This factor is 0 whenever ω is not a multiple of $2\pi/T$. Formally, the following identity holds:

$$\hat{d}(\omega)T = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right) . \tag{15}$$

This can be seen by combining the equations in (5) with $\omega = 2\pi i n/T$:

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt' \, e^{2\pi i n t'/T} f(t') e^{-2\pi i n t/T} \\ &= \frac{1}{T} \int_{-T/2}^{T/2} dt' \sum_{n=-\infty}^{\infty} e^{2\pi i n (t'-t)/T} f(t') . \end{aligned} \tag{16}$$

From this expression, we see that, formally,

$$\sum_{n=-\infty}^{\infty} e^{2\pi i n (t'-t)/T} = T \sum_{-\infty}^{\infty} \delta\left(t' - t - nT\right) \tag{17}$$

because $f(t)$ is periodic with period T.
Replacing $t' - t$ by $\omega T^2/2\pi$,

$$\begin{aligned} \hat{d}(\omega)T &= \frac{T}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{i\omega nT} \\ &= \frac{T^2}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \delta\left(\frac{\omega T^2}{2\pi} - nT\right) \\ &= \sqrt{2\pi} \sum_{-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right) . \end{aligned} \tag{18}$$

Problem 27

First, let's find out the solution for the homogeneous equation

$$\frac{d^2 y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - l(l+1)x^2 y = 0 . \tag{19}$$

Let $y = u/x$. Then the differential equation (19) simplifies:

$$\frac{d^2 u}{dx^2} - \frac{l(l+1)}{x^2} u = 0 \tag{20}$$

The form of the equation suggests that possible solutions be of the form $u = x^k$. Plugging this into (20),

$$\begin{gathered} k(k-1)u - l(l+1)u = 0 \\ k = -l, l+1 \; . \end{gathered} \tag{21}$$

Except at $x = a$, y must be a linear combination of x^l and $1/x^{l+1}$. Since the boundary conditions are $y(0) = y(\infty) = 0$,

$$y = \begin{cases} Ax^l & \text{if } x < a \\ B\frac{1}{x^{l+1}} & \text{if } x > a \end{cases} \; . \tag{22}$$

The inhomogeneous equation corresponding to (19) has a delta function in the RHS. Integrating (19) over a small interval $(a-\epsilon, a+\epsilon)$,

$$\left.\frac{dy}{dx}\right|_{a+\epsilon} - \left.\frac{dy}{dx}\right|_{a-\epsilon} = 1 \; . \tag{23}$$

This, together with the continuity of y at $x = a$, gives

$$\begin{aligned} Aa^l &= \frac{B}{a^{l+1}} \\ -(l+1)\frac{B}{a^{l+2}} - Ala^{l-1} &= 1 \; . \end{aligned} \tag{24}$$

Solving for A and B, we get

$$\begin{aligned} A &= -\frac{a^{-l+1}}{2l+1} \\ B &= -\frac{a^{l+2}}{2l+1} \; . \end{aligned} \tag{25}$$

So,

$$y = \begin{cases} -\frac{a^{1-l}}{2l+1}x^l & \text{if } x < a \\ -\frac{a^{l+2}}{2l+1}\frac{1}{x^{l+1}} & \text{if } x > a \end{cases} \; . \tag{26}$$

Problem 28

The differential equation to deal with is

$$x\frac{d^2 f_n}{dx^2} + g(x)\frac{df_n}{dx} + \lambda_n f_n(x) = 0 \tag{27}$$

Note that f_1 is a linear function in x. Then df_1/dx is constant and we see from (27) that $g(x)$ is also a linear polynomial in x. Set $g(x) = ax + b$. We can change (27) into the Sturm-Liouville form by multiplying by an integration factor:

$$\begin{aligned} e^{\int dx \frac{ax+b}{x}} \left(x \frac{d^2 f_n}{dx^2} + (ax+b) \frac{df_n}{dx} + \lambda_n f_n(x) \right) &= 0 \\ x \left(\frac{d}{dx} e^{\int dx \frac{ax+b}{x}} \frac{df_n}{dx} + e^{\int dx \frac{ax+b}{x}} \frac{1}{x} \lambda_n f_n(x) \right) &= 0 \,. \end{aligned} \tag{28}$$

Since $\int dx \frac{ax+b}{x} = ax + b \log x$ up to a constant,

$$\frac{d}{dx} x^b e^{ax} \frac{df_n}{dx} + x^{b-1} e^{ax} \lambda_n f_n(x) = 0 \,. \tag{29}$$

The weight function is $x^{b-1} e^{ax}$. We want this to be e^{-x}. Therefore, set $b = 1$ and $a = -1$. That is,

$$g(x) = 1 - x \,. \tag{30}$$

To find out the eigenvalue λ_n, notice first that $f_n(x)$ is a polynomial of degree n. Hence the first term of (27) is of degree $n-1$. The second and third terms have degree n. Using $g(x) = 1-x$, the n-th order terms are ($f_n = x^n$+lower order terms)

$$(-n + \lambda_n) x^n = 0. \tag{31}$$

Therefore, $\lambda_n = n$. The final result is

$$x \frac{d^2 f_n}{dx^2} + (1-x) \frac{df_n}{dx} + n f_n(x) = 0 \,. \tag{32}$$

Problem 29

(a)

Differentiating the generating function with respect to h,

$$\begin{aligned} \frac{d}{dh} e^{2hx - h^2} &= \frac{d}{dh} \sum_{n=0}^{\infty} H_n(x) \frac{h^n}{n!} \\ (2x - 2h)\, e^{2hx-h^2} &= \sum_{n=0}^{\infty} H_n(x) \frac{n h^{n-1}}{n!} \\ \sum_{n=0}^{\infty} (2x - 2h)\, H_n(x) \frac{h^n}{n!} &= \sum_{n=0}^{\infty} H_{n+1}(x) \frac{h^n}{n!} \\ \sum_{n=0}^{\infty} 2x H_n(x) \frac{h^n}{n!} - \sum_{n=1}^{\infty} 2n H_n(x) \frac{h^n}{n!} &= \sum_{n=0}^{\infty} H_{n+1}(x) \frac{h^n}{n!} \,. \end{aligned} \tag{33}$$

Hence

$$\begin{aligned} H_1(x) &= 2xH_0(x) \\ H_{n+1}(x) &= 2xH_n(x) - 2nH_{n-1}(x) \end{aligned} \tag{34}$$

(b)

Note that

$$\begin{aligned} \int_{-\infty}^{\infty} dx \exp\left(-\frac{x^2}{2} + 2hx - h^2\right) &= \int_{-\infty}^{\infty} dx \exp\left[-\frac{1}{2}(x-2h)^2 + h^2\right] \\ &= \sqrt{2\pi} e^{h^2} . \end{aligned} \tag{35}$$

Using the generating function,

$$\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dx e^{-x^2/2} H_n(x) \frac{h^n}{n!} = \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{h^{2n}}{n!}. \tag{36}$$

Comparing term by term,

$$\sum_{-\infty}^{\infty} e^{-x^2/2} H_n(x) dx = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sqrt{2\pi} \frac{n!}{(n/2)!} & \text{if } n \text{ even} . \end{cases} \tag{37}$$