Ph129 PS 8 solutions

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Problem 35

First, consider the homogeneous equation

$$\nabla^2 G(\mathbf{x},\mathbf{y}) + k^2 G(\mathbf{x},\mathbf{y})) = 0\ , \tag{1}$$

where y is fixed and x varies.

In the spherical coordinate system where $\mathbf{x} = (r,\theta,\phi)$,

$$\begin{aligned} \frac{1}{r}\frac{\partial^2}{\partial r^2}(rG) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 G}{\partial\phi^2} + k^2 G(x,y) = 0 \\ \frac{1}{r}\frac{\partial^2}{\partial r^2}(rG) - \frac{L^2}{r^2}G + k^2 G = 0\ , \end{aligned} \tag{2}$$

where

$$L^2 = -\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2 G}{\partial\phi^2}$$

is the angular momentum operator. We know that the eigenvalue of L^2 is given by $l(l+1),\ l = 1,2,\cdots$, and the corresponding eigenfunctions are the spherical harmonics $Y_{lm}(\theta,\phi)$ where

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi},\ l = 0,1,\cdots,m = -l,\cdots,l. \tag{3}$$

The spherical harmonics satisfy the completeness relation

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) = \delta(\phi'-\phi)\delta(\cos\theta-\cos\theta')\ . \tag{4}$$

Assuming a solution of the form $G = R(r)Y_{lm}(\theta, \phi)/\sqrt{r}$ for (2), $R(r)$ satisfies

$$\frac{1}{r}\frac{d^2}{dr^2}(rR) - \frac{l(l+1)}{r^2}R + k^2R = 0 \; , \tag{5}$$

$R(r)$ is a linear combination of the spherical Bessel functions

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+1/2}(x), \qquad n_l(x) = \sqrt{\frac{\pi}{2x}}Y_{l+1/2}(x) \; , \tag{6}$$

where J_n and Y_n are the Bessel functions.

Set $\mathbf{y} = (r', \theta', \phi')$. Let's consider the Helmholtz equation with a delta function on the RHS:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rG(\mathbf{x}, \mathbf{y})) - \frac{L^2}{r^2}G(\mathbf{x}, \mathbf{y}) + k^2G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x}, \mathbf{y}) \; , \tag{7}$$

The delta function in the spherical coordinate systems has the expression

$$\delta(\mathbf{x}, \mathbf{y}) = \frac{1}{r^2}\delta(r - r')\delta(\phi - \phi')\delta(\cos\theta - \cos\theta') \; . \tag{8}$$

The quickest way is to see why it does is to integrate both sides over some volume element.

Note that the completeness relation (4) says

$$\delta(\mathbf{x}, \mathbf{y}) = \frac{1}{r^2}\delta(r - r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi) \; . \tag{9}$$

As we have seen already, the solution to the Helmholtz equation can be expanded in terms of the spherical harmonics since the differential operator of the Helmholtz equation is rotationally invariant. Hence

$$G(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm}(r, r', \theta', \phi')Y_{lm}(\theta, \phi) \; . \tag{10}$$

Using (9) and (10), (7) can be rewritten as

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rC_{lm}) - \frac{l(l+1)}{r^2}C_{lm} + k^2C_{lm}\right]Y_{lm}(\theta, \phi) \\ = \frac{1}{r^2}\delta(r - r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi) \; .\end{aligned} \tag{11}$$

This implies

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(rC_{lm}\right)-\frac{l(l+1)}{r^2}C_{lm}+k^2C_{lm}=\frac{1}{r^2}\delta(r-r')Y^*_{lm}(\theta',\phi')\ . \tag{12}$$

Since $C_{lm}=C_{lm}(r,r',\theta',\phi')$, we expect that this factorizes into $C_{lm}=f_l(r,r')Y^*_{lm}(\theta',\phi')$. Then $f(r,r')$ satisfies

$$\frac{1}{r}\frac{d^2}{dr^2}\left(rf_l(r,r')\right)-\frac{l(l+1)}{r^2}f_l(r,r')+k^2f_l(r,r')=\frac{1}{r^2}\delta(r-r')\ . \tag{13}$$

When $r\neq r'$, the solution just satisfies the same equation as (5). Since we want the solution to vanish at $r=a$ and expect no singularity at $r=0$,

$$f_l(r,r')=\begin{cases}Aj_l(kr) & \text{if } r<r'\\ B\left(j_l(kr)n_l(ka)-j_l(ka)n_l(kr)\right) & \text{if } r>r'\end{cases}\ . \tag{14}$$

The boundary conditions are satisfied at $r=0$ and a. We require f_l satisfy the following two relations to ensure the the delta function singularity when differentiated twice:

$$\begin{gathered}f_l(r'-\epsilon)=f_l(r'+\epsilon)\\ \frac{d}{dr}\left(rf_l\right)\Big|_{r'+\epsilon}-\frac{d}{dr}\left(rf_l\right)\Big|_{r'-\epsilon}=\frac{1}{r'}\end{gathered} \tag{15}$$

as $\epsilon\to 0$. Then

$$\begin{aligned}B(j_l(kr')n_l(ka)-j_l(ka)n_l(kr'))&=Aj_l(kr')\\ B(j'_l(kr')n_l(ka)-j_l(ka)n'_l(kr'))-Aj'_l(kr')&=\frac{1}{kr'^2}\ .\end{aligned} \tag{16}$$

We have the following relation for the spherical Bessel functions:

$$j'_l(x)n_l(x)-j_l(x)n'_l(x)=-\frac{1}{x^2}\ . \tag{17}$$

Using this, we can simplify (16) and solve for A and B. The result is

$$\begin{aligned}A&=\frac{k}{j_l(ka)}\left(j_l(ka)n_l(kr')-j_l(kr')n_l(ka)\right)\\ B&=-k\frac{j_l(kr')}{j_l(ka)}\ .\end{aligned} \tag{18}$$

So $f_l(r,r')$ is given by

$$\begin{aligned}f_l(r,r')&=\begin{cases}\frac{k}{j_l(ka)}\left(j_l(ka)n_l(kr')-j_l(kr')n_l(ka)\right)j_l(kr) & (r<r')\\ -k\frac{j_l(kr')}{j_l(ka)}\left(j_l(kr)n_l(ka)-j_l(ka)n_l(kr)\right) & (r>r')\end{cases}\\ &=k\frac{j_l(kr_<)}{j_l(ka)}\left(j_l(ka)n_l(kr_>)-j_l(kr_>)n_l(ka)\right)\ .\end{aligned} \tag{19}$$

Here $r_>$ denotes $\max(r, r')$ and $r_<$ $\min(r, r')$.
Finally, the Green's function is

$$\begin{aligned}G(\mathbf{x}, \mathbf{y}) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm}(r, r', \theta, \phi')Y_{lm}(\theta, \phi)\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l(r, r')Y^*_{lm}(\theta, \phi')Y_{lm}(\theta, \phi)\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} k\frac{j_l(kr_<)}{j_l(ka)}\left(j_l(ka)n_l(kr_>) - j_l(kr_>)n_l(ka)\right)Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi)\ .\end{aligned} \tag{20}$$

Notice that the Green's function should depend only on r, r' and the angle ψ between $\mathbf{x}$ and $\mathbf{y}$. Hence we have the following expansion for the completeness relation (4):

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi) &= \sum_{l=0}^{\infty} C_l P_l(\cos\psi)\\ \delta(\phi' - \phi)\delta(\cos\theta - \cos\theta') &= \sum_{l=0}^{\infty} C_l P_l(\cos\psi)\ .\end{aligned} \tag{21}$$

We may set the coordinate system in such a way that $\theta' = 0$. Then $\theta = \psi$. Integrating over ψ and ϕ,

$$\begin{aligned}C_l &= \frac{1}{2\pi}\frac{2l+1}{2}\int_{-1}^{1} d\cos\psi\,\delta(\cos\psi - 1)P_l(\cos\psi)\\ &= \frac{2l+1}{4\pi}\ .\end{aligned} \tag{22}$$

Actually, this integral is ambiguous since the range of the integral does not have $+1$ in its interior. But notice that we have to set ψ' slightly greater than 0 to make $\delta(\phi' - \phi)$ sensible.
Using this, we obtain

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi) = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\cos\psi)\ . \tag{23}$$

Using this, we can simplify (20 and get

$$G(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{\infty} k\frac{2l+1}{4\pi}P_l(\cos\psi)\frac{j_l(kr_<)}{j_l(ka)}\left[j_l(ka)n_l(kr_>) - j_l(kr_>)n_l(ka)\right]\ . \tag{24}$$

Problem 36

(a)

We wish to solve the differential equation

$$\frac{d^4}{dx^4}G(x,y)+\frac{d^2}{dx^2}G(x,y)=\delta(x-y)\ . \tag{25}$$

To this end, first consider the homogeneous equation

$$\frac{d^4}{dx^4}f(x)+\frac{d^2}{dx^2}f(x)=0\ . \tag{26}$$

We can treat this as a second order differential equation for the function $f''(x)$. Then

$$f''(x)=-A\cos x-B\sin x \tag{27}$$

for some constants A and B. Integrating twice, we obtain

$$f(x)=A\cos x+B\sin x+Cx+D\ . \tag{28}$$

The Green's function should be of this form when $x\neq y$. To get the expression for the Green's function, consider $y>0$ case first. When $x<y$, $G(x,y)$ satisfies the boundary condition

$$G(x,y)=\frac{d}{dx}G(x,y)=\frac{d^2}{dx^2}G(x,y)=\frac{d^3}{dx^3}G(x,y)=0 \qquad \text{when } x=0\ . \tag{29}$$

This means $G(x,y)=0$ for $x<y$. For $x>y$, we assume $G(x,y)$ has the form

$$G(x,y)=A\cos x+B\sin x+Cx+D\ . \tag{30}$$

We get the conditions for $G(x,y)$ at the junction $x=y$ by looking at the equation (25). We see that $G(x,y)$ should yield a delta function when differentiated four times. Hence $G'''(x,y)$(differentiated with respect to x) is discontinuous at $x=y$ with the condition

$$G'''(y+\epsilon)-G'''(y-\epsilon)=1 \qquad \text{as } \epsilon\to 0\ . \tag{31}$$

$G''(x,y)$, $G'(x,y)$ and $G(x,y)$ are continuous at $x=y$ and so they all vanish at $x=y$. Then we can determine the variables A, B, C and D in (30) in a straightforward way and the result is

$$G(x,y)=-\sin(x-y)+x-y \qquad \text{for } x>y>0\ . \tag{32}$$

For $y < 0$, we can proceed in a similar way. In this case $G(x, y) = 0$ when $x > y$. If $x < y$, we also assume the Green's function is of the form (30) and a moment's thought will reveal that

$$G(x, y) = \sin(x - y) - x + y \qquad \text{for } x < y < 0 \tag{33}$$

is the solution.

In sum, for $y > 0$,

$$G(x, y) = \theta(x - y)(-\sin(x - y) + x - y) \tag{34}$$

and, for $y < 0$,

$$G(x, y) = \theta(y - x)(\sin(x - y) - x + y) \tag{35}$$

(b)

Using the Green's function,

$$\begin{aligned} u(x) &= \int_0^\infty G(x, y)\left(-e^{-y}\right) dy \\ &= \int_0^\infty \theta(x - y)(-\sin(x - y) + x - y)\left(-e^{-y}\right) dy \\ &= \int_0^x (\sin(x - y) - x + y)e^{-y} dy \\ &= \frac{1}{2}\left(2 - e^{-x} - 2x - \cos x + \sin x\right) \ . \end{aligned} \tag{36}$$

Notice that the differential equation can be solved in a more direct way by noticing that $u(x) = -e^{-x}/2$ is a particular solution for this differential equation without the proper boundary conditions. To satisfy the boundary conditions, we add a homogenous solution:

$$u(x) = -\frac{1}{2}e^{-x} + A\cos x + B\sin x + Cx + D \ . \tag{37}$$

Now we impose the boundary conditions $u(0) = u'(0) = u''(0) = u'''(0) = 0$ and this determines all undetermined variables and

$$u(x) = \frac{1}{2}\left(-e^{-x} - \cos x + \sin x - 2x + 2\right) \ , \tag{38}$$

which agrees the solution derived by the Green's function method.

Problem 37

A wave function $\psi(x)$ evolves into another wave function $\psi'(x)$ after time t has been elapsed where

$$\psi'(x) = \int_{-\infty}^{\infty} dy\, U(x,y;t)\psi(y) \ . \tag{39}$$

If we transform $\psi'(x)$ by time $-t$, we get a wave function $\psi''(x)$ where

$$\psi''(x) = \int_{-\infty}^{\infty} dy\, U(x,y;-t)\psi'(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dydz\, U(x,y;-t)U(y,z;t)\psi(z) \ . \tag{40}$$

What we would like to verify is if the expression

$$\int_{-\infty}^{\infty} dy\, U(x,y;-t)U(y,z;t) \tag{41}$$

is equivalent to $\delta(x-z)$. Let's check this:

$$\begin{aligned}
&\int_{-\infty}^{\infty} dy\, U(x,y;-t)U(y,z;t) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} dy \left(1 + i\frac{t}{|t|}\right)\left(1 - i\frac{t}{|t|}\right)\frac{m}{2\pi|t|}\exp\left[-\frac{im(x-y)^2}{2t}\right]\exp\left[\frac{im(y-z)^2}{2t}\right] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} dy\, 2\frac{m}{2\pi|t|}\exp\left[\frac{im}{2t}(x^2-z^2)\right]\exp\left[\frac{im}{t}y(x-z)\right] \\
&= 2\pi\delta\left(\frac{m}{t}(x-z)\right)\frac{m}{2\pi|t|}\exp\left[\frac{im}{2t}(x^2-z^2)\right] \\
&= \delta\left(x-z\right) \ .
\end{aligned} \tag{42}$$

Here I used the following properties of the delta function:

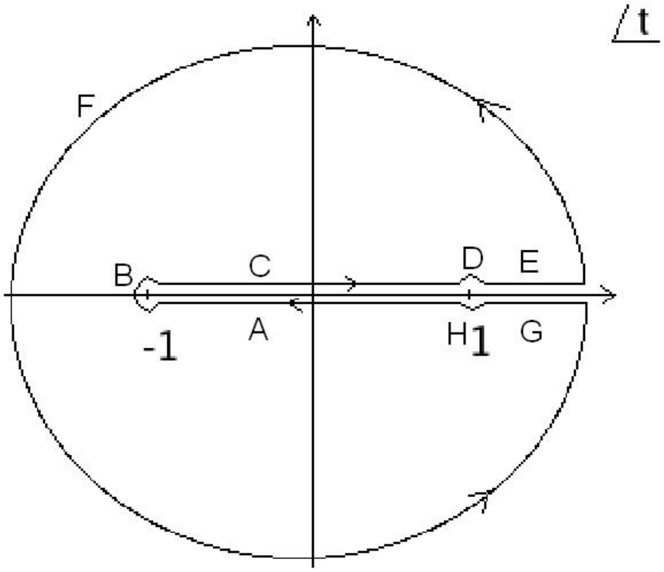
$$f(x)\delta(x) = f(0)\delta(x) \qquad \text{and} \qquad \delta(ax) = \frac{1}{|a|}\delta(x) \ . \tag{43}$$

Problem 38

We start with

$$Q_n(z) = \frac{1}{2}P_n(z)\ln\left(\frac{z+1}{z-1}\right) + f_{n-1}(z) \tag{44}$$

Take the following contour to apply Cauchy's formula:



Then Cauchy's formula says

$$Q_n(z) = \frac{1}{2\pi i} \oint \frac{Q_n(t)}{t-z}\, dt \ . \tag{45}$$

Notice that (44) has a discontinuity

$$Q_n(x+i\epsilon) - Q_n(x-i\epsilon) = -i\pi P_n(x) \tag{46}$$

for $-1 < x < 1$: If we assume that $\ln(z+1)$ and $\ln(z-1)$ both has a branch cut along the positive real axis, when we encircle around -1 from $x+i\epsilon$ to $x-i\epsilon$, $\ln(z+1)$ gets an additive term $2\pi i$. Integration along F is zero if we push the circle to infinity since $Q_n(z) \backsim z^{-(n+1)}$. Integration along E and G cancel. Integration along B, D and H tend to 0 as we take those contours smaller and smaller. So we end up with

$$\begin{aligned}
Q_n(z) &= \frac{1}{2\pi i}\left(\int_A + \int_C\right)\frac{Q_n(t)}{t-z}\,dt \\
&= \frac{1}{2\pi i}\int_{-1}^{1} -\left(\frac{Q_n(t-i\epsilon)}{t-i\epsilon-z}+\frac{Q_n(t+i\epsilon)}{t+i\epsilon-z}\right)\,dt \\
&= \frac{1}{2\pi i}\int_{-1}^{1}\frac{\left[-Q_n(t-i\epsilon)+Q_n(t+i\epsilon)\right](t-z)-i\epsilon\left[Q_n(t-i\epsilon)+Q_n(t+i\epsilon)\right]}{(t-z)^2+\epsilon^2}\,dt \\
&= \frac{1}{2\pi i}\int_{-1}^{1}\frac{Q_n(t+i\epsilon)-Q_n(t-i\epsilon)}{t-z}\,dt \qquad \text{as } \epsilon\to 0 \\
&= -\frac{1}{2\pi i}\int_{-1}^{1}\frac{i\pi P_n(x)}{t-z}\,dt \\
&= \frac{1}{2}\int_{-1}^{1}\frac{P_n(x)}{z-t}\,dt
\end{aligned} \tag{47}$$

This problem can be solved by using another method. First we will derive a useful formula when x is real:

$$\mathrm{Im}\frac{1}{x-i\epsilon} = \pi\delta(x) \qquad \text{where } \epsilon\to +0\ . \tag{48}$$

To see this, notice that, given a real function $f(x)$,

$$2i\int_{-\infty}^{\infty}\mathrm{Im}\frac{f(x)}{x-i\epsilon}dx = \int_{-\infty}^{\infty}\frac{f(x)}{x-i\epsilon}dx - \int_{-\infty}^{\infty}\frac{f(x)}{x+i\epsilon}dx\ . \tag{49}$$

Now, we consider the first integral as a complex one and move the point $+i\epsilon$ a little bit downward so that it lies at the origin. To keep the integration contour passing below the singular point, the contour has to be deformed in such a way that it now takes a half circle around the origin. Similarly, we move the point $-i\epsilon$ to the origin in the second integral and make the contour to half-way encircle the origin. The result is that we take a contour integral along a tiny circle around the origin. By the residue theorem,

$$\int_{-\infty}^{\infty}\mathrm{Im}\frac{f(x)}{x-i\epsilon}dx = \frac{1}{2i}\oint\frac{f(x)}{x}dx = \pi f(0)\ . \tag{50}$$

Since the RHS of (48) gives $\pi f(0)$ when multiplied by $f(x)$ and then integrated, we prove (48).

Note that

$$Q_n'(z) = \frac{1}{2}\int_{-1}^{+1}\frac{P_n(t)dt}{z-t} \tag{51}$$

has a branch cut between -1 and 1 and analytic(holomorphic) except this cut. Using (48), we can get the discontinuity of $Q'_n(z)$ at $z = x$ for $-1 < x < 1$.

$$\begin{aligned} Q'_n(x+i\epsilon) - Q'_n(x-i\epsilon) &= \frac{1}{2}\left[\int_{-1}^{+1} \frac{P_n(t)dt}{x+i\epsilon-t} - \int_{-1}^{+1} \frac{P_n(t)dt}{x-i\epsilon-t}\right] \\ &= -i\,\mathrm{Im}\int_{-1}^{+1} \frac{P_n(t)dt}{x-t-i\epsilon} \\ &= -i\int_{-1}^{+1} P_n(t)\pi\delta(x-t)dt \\ &= -i\pi P_n(x)\ . \end{aligned} \tag{52}$$

Notice that $Q_n(z)$ in (44) has the same discontinuity as $Q'_n(z)$.
Both (51) and (44) tends to 0 as $z \to \infty$. Hence their difference $Q'_n(z) - Q_n(z)$ also should tend to 0 as $z \to 0$. Moreover, the difference does not have a branch cut since we subtracted the same branch cut with the same amount of discontinuity. Since neither has a pole, their difference does not have a pole also. Consequently, the difference $Q'_n(z) - Q_n(z)$ is a bounded analytic function with no singularities. Since it tends to 0 as $z \to \infty$, $Q'_n(z) - Q_n(z) = 0$ for all z. Hence the two expressions coincide.