Ph129 PS 9 solutions

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Problem 40

Let's consider $t > t_0 = 0$ first. The wave function at time t is given by

$$\psi(x,t) = \int_{-\infty}^{\infty} U(x,t;x_0,0)\psi(x_0,0)dx_0 \ . \tag{1}$$

Using the explicit expression for the propagator U and the initial wave function $\psi(x_0,0)$,

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}(1-i)\sqrt{\frac{m}{2\pi t}}\left(\frac{1}{\pi a^2}\right)^{1/4} \exp\left[\frac{im(x-x_0)^2}{2t} - \frac{x_0^2}{2a^2} + ip_0x_0\right] dx_0 \ . \tag{2}$$

It is useful to note the following integral formula: For $a > 0$,

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \int_{-\infty}^{\infty} e^{-a\left(x-\frac{b}{2a}\right)^2+\frac{b^2}{4a}}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}} \ . \tag{3}$$

Then the integral in (2) is

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp\left[\frac{im(x-x_0)^2}{2t} - \frac{x_0^2}{2a^2} + ip_0x_0\right] dx_0 \\
&\quad = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{1}{a^2} - \frac{im}{t}\right)x_0^2 + i\left(p_0 - \frac{xm}{t}\right)x_0\right] dx_0 \\
&\quad = \sqrt{\frac{2\pi t a^2}{t - ima^2}} \exp\left[-\frac{a^2 t}{2(t-ima^2)}\left(p_0 - \frac{mx}{t}\right)^2\right] \ .
\end{aligned} \tag{4}$$

Using this to simplify (2), we obtain

$$\psi(x,t) = \frac{1-i}{\sqrt{2}}\left(\frac{1}{\pi a^2}\right)^{1/4}\sqrt{\frac{ma^2}{t-ima^2}} \exp\left[-\frac{a^2m^2}{2t(t-ima^2)}\left(x - \frac{p_0 t}{m}\right)^2\right] \ . \tag{5}$$

The probability density function $|\psi(x,t)|^2$ is then

$$\begin{aligned} |\psi(x,t)|^2 &= \sqrt{\frac{1}{\pi a^2}} \frac{ma^2}{\sqrt{t^2+m^2a^4}} \exp\left[-\frac{a^2m^2}{t^2+m^2a^4}\left(x-\frac{p_0 t}{m}\right)^2\right] \\ &= \frac{1}{\sqrt{2\pi}\sigma(t)} \exp\left[-\frac{1}{2\sigma(t)^2}\left(x-\frac{p_0 t}{m}\right)^2\right] , \end{aligned} \tag{6}$$

where $\sigma(t)$, given by

$$\sigma(t) = \frac{a}{\sqrt{2}}\sqrt{1+\frac{t^2}{m^2a^4}} , \tag{7}$$

is the standard deviation of the probability distribution $|\psi(x,t)|^2$.

The average position of the particle is $x = p_0 t/m$, so the particle really moves at a speed of p_0/m. The standard deviation $\sigma(t)$ indicates that the Gaussian packet gets broadened as time increases.

The case for $t < t_0 = 0$ can be handled in completely the same way except for some sign changes. If we follow the same procedures as above, the $|\psi(x,t)|^2$ and $\sigma(t)$ are given by the same expression as the case for $t > 0$.

Problem 41

This time, we temporarily ignore the boundary conditions and concentrate on the delta function singularity in the inhomogeneous differential equation

$$\left(\nabla^2 + k^2\right) G(\mathbf{x},\mathbf{y}) = \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{8}$$

We assume that the solution is spherically symmetric if $\mathbf{y}$ is put to the origin. Then we try

$$\frac{1}{r}\frac{d^2}{dr^2}\left(rG(r)\right) + k^2 G(r) = \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{9}$$

Here $r = |\mathbf{x}-\mathbf{y}|$. This equation just gives a harmonic solution when $r \neq 0$;

$$\begin{aligned} \frac{d^2}{dr^2}\left(rG(r)\right) + k^2 rG(r) &= 0 \\ G(r) = A\frac{\cos kr}{r} + B\frac{\sin kr}{r} &. \end{aligned} \tag{10}$$

Unlike the one dimensional problems, it is not possible to split the space into two so that each region has its own homogenous solution. Rather, in this case, the single function

$$\frac{\cos kr}{r}$$

has the required singularity at the origin. To see this, we integrate (8) over a small volume containing $\mathbf{y}$. Then the second tern in the LHS vanishes and using Stoke's theorem,

$$\begin{aligned} \int \nabla^2 G(r) d^3\mathbf{x} &= \int \delta^{(3)}(\mathbf{x}-\mathbf{y})\, d^3\mathbf{x} \\ \int \frac{dG(r)}{dr} r^2\, d\Omega &= 1 \\ -kr\left(\sin kr - \cos kr\right) - 4\pi\left(A\cos kr + B\sin kr\right) &= 1 \\ A &= -\frac{1}{4\pi}\ . \end{aligned} \tag{11}$$

Here we take the limit $r \to 0$.
So a particular solution is

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi}\frac{\cos k|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}|} + B\frac{\sin k|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}|} \tag{12}$$

we can make a more specific choice and set $B = 0$ since the second term is smooth all over the space and is just a solution of the homogeneous differential equation. The general solution for the homogeneous equation is of the form

$$g(\mathbf{x},\mathbf{y}) = \sum_{l=0}^{\infty} A_l j_l(k|x|) P_l(\cos\theta) \tag{13}$$

where j_l is the spherical Bessel function and θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Now we add this homogeneous piece to the particular solution

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi}\frac{\cos k|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}|} + g(\mathbf{x},\mathbf{y}) \tag{14}$$

and impose the boundary condition $G(\mathbf{x},\mathbf{y}) = 0$ for $|x| = a$. The boundary condition reads

$$-\frac{1}{4\pi}\frac{\cos k\sqrt{a^2+y^2-2ay\cos\theta}}{\sqrt{a^2+y^2-2ay\cos\theta}} + \sum_{l=0}^{\infty} A_l j_l(ka) P_l(\cos\theta) = 0 \tag{15}$$

where $y = |\mathbf{y}|$. Using the orthogonality relation

$$\int_{-1}^{1} P_l(x) P_k(x) dx = \frac{2}{2l+1}\delta_{lk} \tag{16}$$

we obtain

$$\begin{aligned}\int_{-1}^{1} dz P_l(z) \left(-\frac{1}{4\pi} \frac{\cos k\sqrt{a^2+y^2-2ay\cos z}}{\sqrt{a^2+y^2-2ay\cos z}} \right) + A_l j_l(ka) \frac{2}{2l+1} = 0 \\ A_l = \frac{2l+1}{2} \frac{1}{j_l(ka)} \frac{1}{4\pi} \int_{-1}^{1} dz P_l(z) \frac{\cos k\sqrt{a^2+y^2-2ay\cos z}}{\sqrt{a^2+y^2-2ay\cos z}} \ .\end{aligned} \tag{17}$$

Therefore, the Green's function is

$$\begin{aligned}G(\mathbf{x},\mathbf{y}) = &-\frac{1}{4\pi} \frac{\cos k|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}|} \\ &+ \sum_{l=0}^{\infty} \frac{2l+1}{8\pi} \frac{j_l(k|x|)}{j_l(ka)} P_l(\cos\theta) \int_{-1}^{1} dz P_l(z) \frac{\cos k\sqrt{a^2+y^2-2ay\cos z}}{\sqrt{a^2+y^2-2ay\cos z}} \ .\end{aligned} \tag{18}$$

Problem 42

(a)

We will solve this problem mainly using the resolvents lecture notes. The differential equation is

$$-\frac{1}{2m} \frac{d^2}{dx^2} G(x,y;z) - zG(x,y;z) = \delta(x-y) \ . \tag{19}$$

First consider the homogeneous equation. The solution satisfying the left boundary condition $u(a) = 0$ is

$$u_L(x) = \sin\rho(x-a) \tag{20}$$

and that satisfying the right boundary condition $u(b) = 0$ is

$$u_R(x) = \sin\rho(x-b) \ . \tag{21}$$

Here $\rho = \sqrt{2mz}$.

The Wronskian is $W(z) = \rho \sin\rho(a-b)$. Therefore, we have the Green's function

$$\begin{aligned}G(x,y;z) = \frac{2m}{\rho\sin\rho(a-b)} &(\sin\rho(x-a)\sin\rho(y-b)\theta(y-x) \\ &+ \sin\rho(y-a)\sin\rho(x-b)\theta(x-y)) \ .\end{aligned} \tag{22}$$

(b)

Note that The Green's function (22) is an analytic(holomorphic) function of $z(\rho = \sqrt{2mz})$ except some poles. The poles are the points where the denominator vanishes. Hence $G(x, y; z)$ has poles at

$$\rho(b-a) = k\pi \Leftrightarrow \rho = \frac{k\pi}{b-a} \Leftrightarrow z = \frac{1}{2m}\left(\frac{k\pi}{b-a}\right)^2 . \tag{23}$$

These poles correspond to the eigenvalues of H. Hence, the eigenvalues of H are

$$\omega_k = \frac{1}{2m}\left(\frac{k\pi}{b-a}\right)^2 . \tag{24}$$

The reason why there is such correspondence is explained in the lecture notes. Basically, if you see the Green's function expression in (c), you will notice that each eigenvalue ω_k of H gives a pole to $G(x, y; z)$.

(c)

Perform a contour integration along a small circle around some eigenvalue $z = \omega_k$. From

$$G(x, y; z) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\phi_k^*(y)}{\omega_k - z} \tag{25}$$

this contour integral gives

$$\phi_k(x)\phi_k^*(y) = -\frac{1}{2\pi i}\oint G(x, y; z)dz . \tag{26}$$

To get the residue, examine the behavior of $\sin\rho(a-b)$ near $z = \omega_k$:

$$\begin{aligned} \sin\rho(a-b) &= \sin\rho(a-b) + \frac{d}{dz}\sin\rho(a-b)\Big|_{z=\omega_k}(z-\omega_k) + O((z-\omega_k)^2) \\ &= 0 + (a-b)\cos(\rho(a-b))\frac{d\rho}{dz}\Big|_{z=\omega_k}(z-\omega_k) + O((z-\omega_k)^2) \\ &= (-1)^{k+1}\frac{m}{k\pi}(b-a)^2(z-\omega_k) + O((z-\omega_k)^2) . \end{aligned} \tag{27}$$

Note further that

$$\rho(x-b) = \rho(x-a+a-b) = \rho(x-a) - k\pi . \tag{28}$$

Then

$$\begin{aligned}
\phi_k(x)\phi_k^*(y) &= -\frac{1}{2\pi i}\oint G(x,y;z)dz \\
&= -\lim_{z\to\omega_k}(z-\omega_k)\frac{2m}{\rho\sin\rho(a-b)}(\sin\rho(x-a)\sin\rho(y-b)\theta(y-x) \\
&\qquad\qquad + \sin\rho(y-a)\sin\rho(x-b)\theta(x-y)) \\
&= \frac{2k\pi}{\rho(b-a)^2}(\sin\rho(x-a)\sin\rho(y-a)\theta(y-x)+ \\
&\qquad\qquad \sin\rho(y-a)\sin\rho(x-a)\theta(x-y)) \\
&= \frac{2}{b-a}\sin\rho(x-a)\sin\rho(y-a)\ .
\end{aligned} \tag{29}$$

Here all ρ in the equations are evaluated at $z=\omega_k$. From this we read

$$\phi_k(x) = \sqrt{\frac{2}{b-a}}\sin\rho(x-a)\ . \tag{30}$$

These eigenstates are properly normalized:

$$\int_a^b \phi_k(x)^2 dx = 1\ . \tag{31}$$

(d)

The expression (22) does not have a well-defined limit when ρ is real and a, b tend to $\pm$infinity. To remedy this, we may evaluate $G(x,y;z)$ for imaginary ρ. This is fine because z here is complex and so is ρ. Let $\rho = i\eta$ for real positive η. In the limit $a\to-\infty$ and $b\to\infty$, (22) becomes

$$\begin{aligned}
G(x,y;z) &= \frac{2m}{\eta\sinh\eta(a-b)}(\sinh\eta(x-a)\sinh\eta(y-b)\theta(y-x) \\
&\qquad\qquad + \sinh\eta(y-a)\sinh\eta(x-b)\theta(x-y)) \\
&\to \frac{2m}{-\eta e^{\eta(a-b)}/2}\left[\frac{e^{\eta(x-a)}}{2}\frac{e^{\eta(y-b)}}{2}\theta(y-x)+\frac{e^{\eta(y-a)}}{2}\frac{e^{\eta(x-b)}}{2}\theta(x-y)\right] \\
&= \frac{1}{\eta}e^{-\eta|x-y|} \\
&= i\sqrt{\frac{m}{2z}}e^{i\rho|x-y|}\ .
\end{aligned} \tag{32}$$

What we have verified is the equality of the two expressions on the positive imaginary axis. But the two are analytic functions of z except some sigularities, so they have to equal on all complex z plane.

Problem 43

Let's vary the action,

$$\delta \int L d^3x dt = \int \left(\frac{\partial \phi}{\partial t} \frac{\partial \delta\phi}{\partial t} - \frac{\partial \phi}{\partial x^i} \frac{\partial \delta\phi}{\partial x^i} - \mu^2 \phi \delta\phi \right) d^3x dt \tag{33}$$

Here x^i denote the spatial coordinates x, y, z. Summation over the three indices is implied. Using integration by parts,

$$\begin{aligned}\delta \int L d^3x dt = \int \left(-\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial (x^i)^2} - \mu^2 \phi \right) \delta\phi d^3x dt \\ + \int \left[\frac{\partial}{\partial t} \left(\frac{\partial \phi}{\partial t} \delta\phi \right) - \frac{\partial}{\partial x^i} \left(\frac{\partial \phi}{\partial x^i} \delta\phi \right) \right] d^3x dt \ .\end{aligned} \tag{34}$$

Note that we vary the field only in the interior of the integration region. The field at the surface of the integration region does not change. If the integration region is convex the second term vanishes because, for example,

$$\int \frac{\partial}{\partial t} \left(\frac{\partial \phi}{\partial t} \delta\phi \right) d^3x dt = \int\limits_{space} \left[\frac{\partial \phi}{\partial t} \delta\phi \right]_{t=t_i(x^1,x^2,x^3)}^{t=t_f(x^1,x^2,x^3)} d^3x \tag{35}$$

where t_i and t_f denotes the minimum and maximum of the time coordinates of the points whose spatial coordinates are (x, y, z) , respectively. In the same way, other terms in the second integration vanish. Even though the integration region is not convex, the second term converts to a surface integral using the generalized Stoke's law:

$$\int \left[\frac{\partial}{\partial t} \left(\frac{\partial \phi}{\partial t} \delta\phi \right) - \frac{\partial}{\partial x^i} \left(\frac{\partial \phi}{\partial x^i} \delta\phi \right) \right] d^3x dt = \int\limits_{surface} \frac{\partial \phi}{\partial x^\mu} d^3 S^\mu = 0 \ . \tag{36}$$

Here x^μ stands for the space-time coordinates:$(x^0, x^1, x^2, x^3) = (t, x, y, z)$. dS^μ is a 3-dimensional surface element.

Since the surface term vanishes in (34), the variation of the action is stationary only if

$$-\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial (x^i)^2} - \mu^2 \phi = 0 \ . \tag{37}$$