Ph129b PS 3 solutions

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Problem 10

There are 8 elements in the group C_{4v}. These include the identity (denoted by e), rotations by $\pi/2$ through the perpendicular axis (denoted by r), reflection across the x-axis or y-axis (denoted m_x and m_y) and reflections across the $y = x$ and $y = -x$ (denoted m_p and m_q respectively). It is straightforward to determine the equivalence classes:

$$\{e\}, \quad \{r^2\}, \quad \{r, r^3\}, \quad \{m_x, m_y\} \text{ and } \{m_p, m_q\}. \tag{1}$$

The subgroups (these have possible orders 1,2,4 and 8) are,

$$\begin{aligned} \{e\}, \\ \{e, r^2\}, \quad \{e, m_x\}, \quad \{e, m_y\}, \quad \{e, m_p\}, \quad \{e, m_q\}, \\ \{e, r, r^2, r^3\}, \quad \{e, r^2, m_x, m_y\}, \quad \{e, r^2, m_p, m_q\} \\ C_{4v}. \end{aligned}$$

Those subgroups that are unions of classes are the invariant subgroups, so we have the following invariant subgroups:

$$\{e\}, \quad \{e, r^2\}, \quad \{e, r, r^2, r^3\}, \quad \{e, r^2, m_x, m_y\}, \quad \{e, r^2, m_p, m_q\} \text{ and } C_{4v}. \tag{2}$$

Problem 11

Recall that the multiplication table for the Poincaré group is the following,

$$\Lambda(M_1, z_1)\Lambda(M_2, z_2) = \Lambda(M_1 M_2, M_1 z_1 + z_2). \tag{3}$$

A Lorentz transformation has the form $\Lambda(M, 0)$ and a pure translation has the form $\Lambda(I, z)$. We want to show that any element of the Poincaré group can be written as a pure translation followed by a Lorentz transformation and a Lorentz transformation followed by a pure translation:

$$\Lambda(M, 0)\Lambda(I, M^{-1}z) = \Lambda(M, z) = \Lambda(I, z)\Lambda(M, 0). \tag{4}$$

Problem 12

Let x, y be elements of a vector space V. The averaged scalar product is defined as

$$\{x, y\} = \frac{1}{g} \sum_{a \in G} (D(a)x, D(a)y), \tag{5}$$

where $\{D(a)\}$ is an invertible matrix representation of the group G. A scalar product must satisfy the following four properties:

1. $\{x, x\} \geq 0$ with $\{x, x\} = 0$ iff x = 0.
2. $\{x, y\}^* = \{y, x\}$.
3. $\{x, cy\} = c\{y, x\}$.
4. $\{x_1 + x_2, y\}^* = \{x_1, y\} + \{x_2, y\}$.

We start with a proof of the first property:

$$\{x, x\} = \frac{1}{g} \sum_{a \in G} \sum_{i} |(D(a)x)_i|^2 \geq 0. \tag{6}$$

Now if $x = 0$, clearly every term in the sum is zero and thus $\{x, x\} = 0$. Now suppose $\{x, x\} = 0$, then each term in the sum must vanish. Consider $a = e$, then $D(e)$ is the identity map acting on V. We then have:

$$\{x, x\} = 0 \rightarrow \frac{1}{g} \sum_{i} |(D(e)x)_i|^2 = \frac{1}{g} \sum_{i} |x_i|^2 = 0. \tag{7}$$

Therefore every component of the vector x must be zero. This only works if D is a linear and invertible map. The other properties of the scalar product are trivially satisfied by this definition.

Problem 13

The $2x2$ representation D which performs rotations on vectors (x, y) can be written as:

$$D(\theta) = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \tag{8}$$

We need to find a similarity transformation that diagonalizes this two-dimensional representation into two one-dimensional representations. Consider the following:

$$D'(\theta) = S^\dagger D(\theta) S = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \text{where} \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{9}$$

S can be most easily found by finding the eigenvalues and eigenvectors of the matrix $D(\theta)$. We see that the representation has been reduced to two one-dimensional representations and since the similarity transformation is independent of θ the resulting representations are faithful.

Problem 14

The group D_3 has these elements $\{e, r, r^2, p, pr, pr^2\}$, where r is a rotation by $2\pi/3$ about the primary three-fold axis and p is a rotation by π about one secondary two-fold axis. We want a two-dimensional representation acting on vectors (x, y). We can use $D(\theta)$ as in the previous problem. If the secondary axis is the y-axis, the $x \to -x$ and $y \to y$. Thus we have the following matrix-representation:

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(p) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(r) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}. \tag{10}$$

The remaining matrices can be found through matrix multiplication. One should note that this representation is not(!) reducible since one must perform the same similarity transformation found in the previous problem to all(!) matrices. This results in the following:

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(p) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad D(r) = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{pmatrix}. \tag{11}$$

We see that this is the same irreducible two-dimensional representation in another basis.