ph129b PS 5 solutions

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Problem ~~20-23~~: the Tetrahedron

I graded this problem as one big problem, out of 30 points. Congratulations to everyone that completed it — it is a challenging problem that tied together basically everything we learned about finite groups.

We can simplify the trigonometry by embedding the tetrahedron in a cube, with verteces $v_1 = (0, 0, 0)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 0, 1)$, and $v_4 = (0, 1, 1)$. The potential energy is

$$V = \frac{k}{2}\left\{\left[\frac{1}{\sqrt{2}}(x_2 - x_1) + \frac{1}{\sqrt{2}}(y_2 - y_1)\right]^2 \right. \tag{1}$$

$$+ \left[\frac{1}{\sqrt{2}}(x_3 - x_1) + \frac{1}{\sqrt{2}}(z_3 - z_1)\right]^2 \tag{2}$$

$$+ \left[\frac{1}{\sqrt{2}}(y_4 - y_1) + \frac{1}{\sqrt{2}}(z_4 - z_1)\right]^2 \tag{3}$$

$$+ \left[\frac{1}{\sqrt{2}}(y_2 - y_3) + \frac{1}{\sqrt{2}}(z_3 - z_2)\right]^2 \tag{4}$$

$$+ \left[\frac{1}{\sqrt{2}}(x_2 - x_4) + \frac{1}{\sqrt{2}}(z_4 - z_2)\right]^2 \tag{5}$$

$$\left. + \left[\frac{1}{\sqrt{2}}(x_3 - x_4) + \frac{1}{\sqrt{2}}(y_4 - y_3)\right]^2\right\} . \tag{6}$$

Each of the six terms corresponds to one spring — for example, the first term, which involves coordinates with subscripts 1 and 2, corresponds to the spring connecting v_1 and v_2. The relative signs of each coordinate inside the square brackets is chosen as follows: increasing a positive sign coordinate causes that particular spring to stretch, increasing a negative sign coordinate causes that spring to compress. Analogous to what was done in class, we can write

$$L = T - V = \frac{m}{2}\sum_{i=1}^{12}\dot{\eta}_i^2 - \frac{k}{2}\sum_{i,j=1}^{12}\eta_i U_{ij}\eta_j \tag{7}$$

where $\eta = \{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, x_4, y_4, z_4\}$ and

$$U = \frac{1}{2}\left(\begin{array}{ccc|ccc|ccc|ccc}
2 & 1 & 1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
1 & 2 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 2 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\
\hline
-1 & -1 & 0 & 2 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\
-1 & -1 & 0 & 1 & 2 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 2 & 0 & 1 & -1 & 1 & 0 & -1 \\
\hline
-1 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 2 & -1 & 1 & -1 & 0 \\
-1 & 0 & -1 & 0 & 1 & -1 & 1 & -1 & 2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 2 & -1 & -1 \\
0 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 & 1 \\
0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 2
\end{array}\right) . \tag{8}$$

The Euler-Lagrange equation is

$$m\ddot{\eta}_i + k\sum_j U_{ij}\eta_j = 0 . \tag{9}$$

Assuming oscillatory solutions $\eta(t) = \xi e^{i\omega t}$, we get the matrix equation

$$\left(U - \frac{m\omega^2}{k} I\right)\xi = 0 . \tag{10}$$

Thus we can solve the problem by solving a 12×12 eigenvalue problem. The eigenvectors ξ are the normal modes of the system, and the eigenvalues λ are related to the normal frequencies by $\omega^2 = k\lambda/m$.

Now we want to attack the problem with our group theory methods. First, let's recall the character table for T_d, the tetrahedral symmetry group.

N_k	$l_k \rightarrow$ C_k	1 $\chi^{(1)}$	1 $\chi^{(2)}$	2 $\chi^{(3)}$	3 $\chi^{(4)}$	3 $\chi^{(5)}$
1	$C_1 = \{e\}$	1	1	2	3	3
6	$C_2 = \{(ab)\}$	1	-1	0	-1	1
8	$C_3 = \{(abc)\}$	1	1	-1	0	0
6	$C_4 = \{(abcd)\}$	1	-1	0	1	-1
3	$C_5 = \{(ab)(cd)\}$	1	1	2	-1	-1

Next, for an element $g \in T_d$, we want to consider a 12×12 representation $D(g)$ which acts on our 12-dimensional vector space η. In particular, we want to decompose representation D into irreps, and so we need to find the character table of D. Clearly, $D(e) = I_{12}$, so $\chi(e) = 12$. The element $p = (34)$ reflects the tetrahedron across a mirror plane containing the line $y = x$ and the $\hat{z}$-axis. Thus, we have

$$D(p) = \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & 0 & P \\ 0 & 0 & P & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{11}$$

The positions of the P's in $D(p)$ are such that masses 3 and 4 are swapped under (34). Under this reflection across the $y = x$ plane, $(x, y, z) \to (y, x, z)$ — and P is the matrix which performs this transformation. Similarly, for $r = (234)$ we have

$$D(r) = \begin{pmatrix} R & 0 & 0 & 0 \\ 0 & 0 & 0 & R \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \tag{12}$$

where R gives the transformation $(x, y, z) \to (z, x, y)$ as a result of rotation r. Element $m = (12)(34)$ performs a rotation on the tetrahedron by π about $\hat{z}$, and so we have

$$D(m) = \begin{pmatrix} 0 & M & 0 & 0 \\ M & 0 & 0 & 0 \\ 0 & 0 & 0 & M \\ 0 & 0 & M & 0 \end{pmatrix} \qquad M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{13}$$

Finally, the element $q = (1234)$ is represented

$$D(q) = \begin{pmatrix} 0 & 0 & 0 & Q \\ Q & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & Q & 0 \end{pmatrix} . \tag{14}$$

We don't need to work out what Q is. Now we can calculate the character table of D:

	$l_k \to$	12
N_k	C_k	χ
1	C_1	12
6	C_2	2
8	C_3	0
6	C_4	0
3	C_5	0

Next, we want to decompose D into irreps $D^{(i)}$. The general decomposition is

$$D = a_1 D^{(1)} \oplus a_2 D^{(2)} \oplus a_3 D^{(3)} \oplus a_4 D^{(4)} \oplus a_5 D^{(5)} . \tag{15}$$

We can find the coefficients a_i by the formula

$$a_i = \frac{1}{24} \sum_{k=1}^{5} N_k \chi^{(i)}(C_k) \chi(C_k) . \tag{16}$$

This gives

$$D = D^{(1)} \oplus D^{(3)} \oplus D^{(4)} \oplus 2D^{(5)} \; . \tag{17}$$

This decomposition tells us that we have 1 doubly-degenerate eigenvalue, and 3 triply-degenerate eigenvalues, for a total of 5 unique eigenvalues. Finally, we want to find a set of trace equations that can be solved for the eigenvalues λ. A general trace equation is

$$\mathrm{Tr}[D(g)U] = \lambda_1 \chi^{(1)}(g) + \lambda_3 \chi^{(3)}(g) + \lambda_4 \chi^{(4)}(g) + (\lambda_{51} + \lambda_{52})\chi^{(5)}(g) \; . \tag{18}$$

On the left side, we can evaluate fully using the matrices we derived above. On the RHS, we know all the $\chi^{(i)}$'s. Thus we have a set of linear equations for λ_i:

$$\mathrm{Tr}[D(e)U] = 12 = \lambda_1 + 2\lambda_3 + 3\lambda_4 + 3(\lambda_{51} + \lambda_{52}) \tag{19}$$
$$\mathrm{Tr}[D(p)U] = 6 = \lambda_1 - \lambda_4 + (\lambda_{51} + \lambda_{52}) \tag{20}$$
$$\mathrm{Tr}[D(r)U] = 3 = \lambda_1 - \lambda_3 \tag{21}$$
$$\mathrm{Tr}[D(m)U] = 4 = \lambda_1 + 2\lambda_3 - \lambda_4 - (\lambda_{51} + \lambda_{52}) \tag{22}$$

The solution to these equations is

$$\lambda_1 = 4, \; \lambda_3 = 1, \; \lambda_4 = 0, \; \lambda_{51} + \lambda_{52} = 2 \; . \tag{23}$$

We could write the corresponding equation for $\mathrm{Tr}[D(q)]$, but this wouldn't help us break the ambiguity in $\lambda_{51}, \lambda_{52}$. But we can also use

$$\mathrm{Tr}[(D(e)U)^2] = 30 = \lambda_1^2 + 2\lambda_3^2 + 3\lambda_4^3 + 3\lambda_{51}^2 + 3\lambda_{52}^2 \tag{24}$$

which tells us that $\lambda_{51} = 2$ and $\lambda_{52} = 0$. Note that we have two triply-degenerate zero modes. Both correspond to degrees of freedom of the tetrahedron which do not stretch or compress the springs — namely, the three translational and three rotational degrees of freedom in which the entire system is moved in unison. We could have written fewer equations by guessing which modes are the zero modes, but it's nice to have this as a check. The normal frequencies are:

$$\omega_1^2 = \frac{4k}{m}, \; \omega_3^2 = \frac{k}{m}, \; \omega_{51}^2 = \frac{2k}{m}, \; \omega_4^2 = \omega_{52}^2 = 0 \; . \tag{25}$$

Like the 2-d example in class, largest frequency corresponds to the trivial irrep.