ph129b PS 7 solutions

Sean Tulin(revised by C. Park)
(minor revision 080306 fcp)

March 6, 2008

Problem 27: Quaternion Group

The quaternion group has 8 elements $\{1, -1, i, -i, j, -j, k, -k\}$. From the multiplication rules, we can form the multiplication table

1	-1	i	$-i$	j	$-j$	k	$-k$
-1	1	$-i$	i	$-j$	j	$-k$	k
i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	i	1	-1	$-k$	k	j	$-j$
j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	j	k	$-k$	1	-1	$-i$	i
k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	k	$-j$	j	i	$-i$	1	-1

. (1)

Using the multiplication table, we see that there are 5 classes, which are $\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}$, and $\{k, -k\}$. Because there are 5 classes and 8 elements, there must be 4 1×1 irreps and 1 2×2 irrep. The character table is

	$l_k \rightarrow$	1	1	2	3	3
N_k	C_k	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	$\chi^{(4)}$	$\chi^{(5)}$
1	$\{1\}$	1	1	1	1	2
1	$\{-1\}$	1	1	1	1	-2
2	$\{i, -i\}$	1	1	-1	-1	0
2	$\{j, -j\}$	1	-1	1	-1	0
2	$\{k, -k\}$	1	-1	-1	1	0

We can obtain the 1×1 irreps by staring at the multiplication table and appealing to the symmetry of (i, j, k). We can get the last column through the orthogonality relations, or by knowing that the 2×2 irrep is

$$D(1) = I_2 \,, \qquad D(i) = -i\sigma^1 \,, \qquad D(j) = -i\sigma^2 \,, \qquad D(k) = -i\sigma^3 \,. \tag{2}$$

The study of quaternions is a rich (and somewhat esoteric) subject applicable to many branches of physics, but I digress. Now we want to compare this with D_4. Both groups have the same character table. However, they are not isomorphic. Because both groups have 8 elements, we can define a 1-1 map. An example of such a map $\phi : Q \to D_4$ is:

$$\phi : 1 \to e, -1 \to r^2, i \to r, -i \to r^3, j \to s, -j \to sr^2 ... \tag{3}$$

where s is reflection about one particular axis, and r is rotation by π. Notice that although $j^2 = -1$, we have

$$\phi(j)\phi(j) = s^2 = e = \phi(1) \ . \tag{4}$$

This means that the map ϕ does not preserve the group multiplication of Q, and so is not an isomorphism. We can consider all possible 1-1 maps by permuting (i, j, k), but it is impossible to construct one that is isomorphic.

Actually, there are only one element of order 2(that is, -1) in the quaternion group while there are 5 of them in D_4. Hence they cannot be isomorphic.

Problem 28

We want to show that $C_{\alpha\beta\gamma}$ is antisymmetric in all indices. We know that the structure constants satisfy the following two conditions:

$$C^{\alpha}_{\beta\gamma} = -C^{\alpha}_{\gamma\beta} \tag{5}$$

$$C^{\delta}_{\alpha\beta}C^{\mu}_{\delta\nu} + C^{\delta}_{\beta\nu}C^{\mu}_{\delta\alpha} + C^{\delta}_{\nu\alpha}C^{\mu}_{\delta\beta} = 0 \tag{6}$$

which follow, respectively, from the facts that the Lie Bracket is antisymmetric and the generators satisfy the Jacobi identity. First, it's easy to see that

$$C_{\alpha\beta\gamma} = g_{\delta\gamma}C^{\delta}_{\alpha\beta} = -g_{\delta\gamma}C^{\delta}_{\beta\alpha} = -C_{\beta\alpha\gamma} \ . \tag{7}$$

Next, we can use the Jacobi identity to show that $C_{\alpha\beta\gamma} = -C_{\alpha\gamma\beta}$. We have

$$\begin{aligned}
C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} &= g_{\delta\gamma}C^{\delta}_{\alpha\beta} + g_{\delta\beta}C^{\delta}_{\alpha\gamma} && (8)\\
&= \left(C^{\mu}_{\delta\nu}C^{\nu}_{\gamma\mu}\right)C^{\delta}_{\alpha\beta} + \left(C^{\mu}_{\delta\nu}C^{\nu}_{\beta\mu}\right)C^{\delta}_{\alpha\gamma} && (9)\\
&= C^{\nu}_{\gamma\mu}\left(-C^{\delta}_{\beta\nu}C^{\mu}_{\delta\alpha} - C^{\delta}_{\nu\alpha}C^{\mu}_{\delta\beta}\right) + C^{\nu}_{\beta\mu}\left(-C^{\delta}_{\gamma\nu}C^{\mu}_{\delta\alpha} - C^{\delta}_{\nu\alpha}C^{\mu}_{\delta\gamma}\right) && (10)\\
&= C^{\mu}_{\alpha\delta}C^{\delta}_{\beta\nu}C^{\nu}_{\gamma\mu} - C^{\delta}_{\alpha\nu}C^{\mu}_{\beta\delta}C^{\nu}_{\gamma\mu} + C^{\mu}_{\alpha\delta}C^{\nu}_{\beta\mu}C^{\delta}_{\gamma\nu} - C^{\delta}_{\alpha\nu}C^{\nu}_{\beta\mu}C^{\mu}_{\gamma\delta} && (11)\\
&= 0 \ , && (12)
\end{aligned}$$

where, in the first step, we use the definition of $g_{\mu\nu}$; in the second step, we use the Jacobi identity; in the third step, we expand the terms and use the antisymmetry property (1); and finally, we see that the expression vanishes by relabeling the dummy indices $\{\delta \to \mu, \mu \to \nu, \nu \to \delta\}$ in the second and fourth terms. Thus, we have

$$C_{\alpha\beta\gamma} = -C_{\alpha\gamma\beta} \ . \tag{13}$$

Using (3) and (9), we can show that $C_{\alpha\beta\gamma}$ is antisymmetric under permutation of any two indices.

A lot of people were confused about the distinction between external indices and dummy indices. This is a very important point because this sort of index manipulation becomes more common as you progress in physics, so you want it to be second nature. Note that there are two types of indices in this problem: repeated (or dummy) indices which are summed over and non-repeated (or external) indices which are not summed over. You are free to label the dummy indices by any symbol you want, but you must choose a unique label or else you will be confused. For example, suppose we want to calculate a vector $\vec{b}$ which is the component of vector $\vec{a}$ along the direction of unit vector $\vec{n}$. That is, $\vec{b} = (\vec{a} \cdot \vec{n})\vec{n}$. In component notation, we have

$$b_i = a_j n_j n_i \ . \tag{14}$$

Equation (10) is unambiguous and demonstrates proper labeling. However, we could also write

$$b_i = a_i n_i n_i \ . \tag{15}$$

First of all, any time you see more than two of the same index, a red flag should pop up in your head. The equation is ambiquous because you do not know which indices should be summed over. Furthermore, by interpretting (11) incorrectly as $b_i = a_i \sum_i n_i n_i$, you can show that $\vec{a} = \vec{b}$, which is clearly not true. So the moral is: indices can be dangerous things, so use them responsibly.

Another way to solve this problem is to use matrix notation to avoid cluttering of indices. Note that the definition of the metric $g_{\mu\nu}$

$$g_{\mu\nu} = C^{\beta}_{\mu\alpha} C^{\alpha}_{\nu\beta}$$

can be rewritten as

$$g_{\mu\nu} = \mathrm{tr}\,(C_\mu C_\nu)$$

if we treat $C^{\alpha}_{\mu\beta}$ as the (α, β) component of a matrix C_μ.
The Jacobi identity in this notation is

$$[C_\alpha, C_\beta] = C^{\mu}_{\alpha\beta} C_\mu \ .$$

By multiplying by C_ν and tracing the above equation, we get

$$\mathrm{tr}\,(C_\nu\,[C_\alpha, C_\beta]) = C^{\mu}_{\alpha\beta} g_{\mu\nu} = C_{\alpha\beta\nu} \ .$$

This is manifestly antisymmetric in all of α, β, ν if we recall the cyclic invariance of a trace.

Problem 29

An element of a Lie group can be written as $U = e^{i\epsilon^\alpha X_\alpha}$. The complex conjugate representation is $U^* = e^{-i\epsilon^\alpha X^*_\alpha}$. For these two representation to be equivalent, there must exist a matrix S such that $SUS^{-1} = U^*$, or, equivalently, $SX_\alpha S^{-1} = -X^*_\alpha$. The 2 rep of SU(2) has $X_\alpha = \sigma_\alpha/2$, and so the generators of the $\bar{2}$ are $-\sigma^*_\alpha/2$, where σ_α are the 2×2 Pauli matrices. Thus we need to find S such that

$$S\sigma_\alpha = -\sigma^*_\alpha S\ . \tag{16}$$

We can assume that $S \in SU(2)$, because S is a similarity transform (i.e. change of basis). So we can write

$$S = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\ , \tag{17}$$

where $a, b \in$C and $|a|^2 + |b|^2 = 1$. If we plug (13) into (12) and use the explicit form for the σ_α's, we can solve the equations for a and b. Or, we can use the well-known identity

$$\sigma_2\sigma_\alpha = -\sigma^*_\alpha\sigma_2 \tag{18}$$

to see that

$$S = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ . \tag{19}$$

I chose $S = i\sigma_2$ and not, say, $S = \sigma_2$ so that $det(S) = 1$, but that's not manditory. The overall phase of S doesn't matter.

Problem 30

First, we have

$$g^{-1}Ag = \sum_{i=1}^{n_a} g^{-1}a_i g = \sum_{j=1}^{n_a} a_j = A\ , \tag{20}$$

where we have noted that $a_j = g^{-1}a_i g$ is equivalent to a_i, and so is contained in the sum A. So, $g^{-1}Ag$ just reorders the sum over the equivalence class and thus gives back A. This means that

$$g^{-1}ABg = g^{-1}Agg^{-1}Bg = AB\ . \tag{21}$$

Next, we can write

$$AB = \sum_{i=1}^{n_a} a_i \sum_{j=1}^{n_b} b_j = \sum_{k=1}^{n_G} s_k c_k \tag{22}$$

where $c_k \in G$, $\sum_k$ sums over the entire group G, and s_k is a number. We are just stating the fact that the product $a_i b_j$ is another element of G, and so we are expanding AB as a linear combination of group elements c_k. The coefficient s_k just counts how many times a particular $a_i b_j$ gives element c_k, and so we must have $s_k = 0, 1, 2, ...$. Now, let's denote the equivalence classes in G by $\gamma_1, \gamma_2, ..., \gamma_{n_c}$. Then we can write (47) as

$$\begin{aligned} AB &= \sum_{c_k \in \gamma_1} s_k c_k + \sum_{c_k \in \gamma_2} s_k c_k + ... + \sum_{c_k \in \gamma_{n_c}} s_k c_k & (23) \\ &= \sum_{C=\gamma_1}^{\gamma_{n_c}} \sum_{c_k \in C} s_k c_k \ . & (24) \end{aligned}$$

The fact that $g^{-1} AB g = AB$ implies that

$$\sum_{c_k \in C} s_k c_k = \sum_{c_k \in C} s_k g^{-1} c_k g = \sum_{c_j \in C} s_k c_j \ , \tag{25}$$

again using the fact that $g^{-1}(\quad)g$ leaves a sum over a class invariant. Thus s_k must be the same for all members of a particular class C, so let's call it s_C. Thus we have

$$AB = \sum_C \sum_{c_k \in C} s_C c_k = \sum_C s_C C \ . \tag{26}$$

Part b follows easily from this result.

Problem 31

The point of this problem is to show that for $X, Y \in \mathcal{L}$, we can write $e^X e^Y = e^Z$ for some $Z \in \mathcal{L}$. If $Z \notin \mathcal{L}$, then the entire structure of Lie groups and Lie algebras, and indeed our entire world, will come crashing down.

Part a

Suppose $[X, Y] = 0$. Then X and Y can be diagonalized in the same basis. That is, there is a matrix S such that $SXS^{-1} = X_D = \text{diag}(x_1, x_2, ..., x_n)$ and $SYS^{-1} = Y_D = \text{diag}(y_1, y_2, ..., y_n)$. Then we have

$$e^X e^Y = S^{-1} S e^X S^{-1} S e^Y S^{-1} S = S^{-1} e^{X_D} e^{Y_D} S = S^{-1} e^{X_D + Y_D} S = e^{X+Y} \ , \tag{27}$$

and so we have $Z = X + Y \in \mathcal{L}$. A linear combination of any two elements of $\mathcal{L}$ is also in $\mathcal{L}$.

Part b

Now suppose that X and Y don't commute. Let's expand the exponentials in Taylor series and keep only terms through quadratic order:

$$\begin{aligned} e^X e^Y &\simeq \left(1 + X + \frac{1}{2!}X^2\right)\left(1 + Y + \frac{1}{2!}Y^2\right) & (28) \\ &\simeq 1 + X + Y + \frac{1}{2}(X^2 + 2XY + Y^2) & (29) \\ &\simeq 1 + (X+Y) + \frac{1}{2}(X+Y)^2 + \frac{1}{2}[X,Y] & (30) \\ &\simeq e^{X+Y+[X,Y]/2} \, . & (31) \end{aligned}$$

As long as X and Y are small enough — i.e. e^X and e^Y close enough to the identity — we have $Z = X + Y + [X,Y]/2$. Recall that $[X,Y] \in \mathcal{L}$, and so $Z \in \mathcal{L}$ as well.

Part c

Part c is where the real business begins. Good job to all of you that perservered through this problem. Let's begin by verifying the three initial claims.

claim 1

Let

$$f(u) = e^{uX} f_0 + \int_0^u dv \, e^{(u-v)X} g(v) \, . \tag{32}$$

By taking the derivative with respect to u, we have

$$\frac{\partial f}{\partial u} = X e^{uX} f_0 + g(u) + X \int_0^u dv \, e^{(u-v)X} g(v) = X f(u) + g(u) \, . \tag{33}$$

Thus $f(u)$ given by (21) solves the differential equation $\frac{\partial f}{\partial u} = X f(u) + g(u)$. This must be the most general solution, since a first order equation has one free parameter — here, it's f_0. If g doesn't depend on u, then

$$\int_0^u dv \, e^{(u-v)X} g = \frac{e^{uX} - 1}{X} g \equiv h(u, X) g \, , \tag{34}$$

and we have

$$f(u) = e^{uX} f_0 + h(u, X) g \, . \tag{35}$$

claim 2

Let $A(u) = e^{uX} Y e^{-uX}$. Taking the derivative with respect to u, we have

$$\frac{\partial A}{\partial u} = X e^{uX} Y e^{-uX} + e^{uX} Y(-X) e^{-uX} = [X, A] = X_c(A) \ . \tag{36}$$

We can use the result of claim 1, by setting $f \to A, X \to X_c$, and $g \to 0$, to see that

$$\begin{aligned} A(1) &= e^X Y e^{-X} = e^{X_c}(Y) = Y + X_c(Y) + \frac{1}{2!} X_c(X_c(Y)) + ... & (37) \\ &= Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + ... & (38) \end{aligned}$$

where I've written in the last two steps just to clarify exactly what is meant by $e^{X_c}(Y)$.

claim 3

Consider

$$B(t, u) \equiv e^{tX(u)} \frac{\partial}{\partial u} e^{-tX(u)} \ . \tag{39}$$

You might think that you can just evaluate $B(t, u)$ easily by acting with $\frac{\partial}{\partial u}$ on $e^{-tX(u)}$, but here is what happens:

$$\begin{aligned} \frac{\partial}{\partial u} e^{-tX(u)} &= \frac{\partial}{\partial u} \left(\sum_{n=0}^{\infty} \frac{(-t)^n X(u)^n}{n!} \right) & (40) \\ &= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (\dot{X} X^{n-1} + X \dot{X} X^{n-2} + ... + X^{n-1} \dot{X}) \ . & (41) \end{aligned}$$

In other words, we get a mess, because $\dot{X} \equiv \partial X / \partial u$ doesn't commute with X. But let's take the derivative with respect to t:

$$\begin{aligned} \frac{\partial B}{\partial t} &= \frac{\partial}{\partial t} \left(e^{tX(u)} \frac{\partial}{\partial u} e^{-tX(u)} \right) & (42) \\ &= X e^{tX(u)} \frac{\partial}{\partial u} e^{-tX(u)} - e^{tX(u)} \frac{\partial}{\partial u} \left(e^{-tX(u)} X \right) & (43) \\ &= X\, B - \dot{X} - B\, X & (44) \\ &= X(u)_c(B) - \dot{X} \ . & (45) \end{aligned}$$

Now we can solve for B by using claim 1, and we have

$$B(t,u) = e^{tX(u)_c}(B(0,u)) - h(t, X(u)_c)(\dot{X}) = -h(t, X(u)_c)(\dot{X}) \tag{46}$$

since $B(0,u) = 0$. Then, by setting $t = 1$, we get

$$B(1,u) = e^{X(u)} \frac{\partial}{\partial u}\, e^{-X(u)} = -h(1, X(u)_c)(\dot{X}) = \frac{1 - e^{X(u)_c}}{X(u)_c}(\dot{X}) \; . \tag{47}$$

Now on with the proof. We want to consider $Z(u)$ such that $e^{uX}e^{Y} = e^{Z(u)}$. Clearly, when $u = 0$, we have $Z = Y$; but ultimately, we want to find $Z(1)$. Using the result of claim 3, we have

$$e^{Z(u)} \frac{\partial}{\partial u}\, e^{-Z(u)} = \frac{1 - e^{Z(u)_c}}{Z(u)_c}(\dot{Z}) \; . \tag{48}$$

Or, we could write

$$e^{Z(u)} \frac{\partial}{\partial u}\, e^{-Z(u)} = e^{uX}e^{Y} \frac{\partial}{\partial u}\, e^{-Y}e^{-uX} = -X \; . \tag{49}$$

Equating these two lines gives

$$\frac{e^{Z(u)_c} - 1}{Z(u)_c}(\dot{Z}) = X \; , \tag{50}$$

or, by inverting,

$$\dot{Z} = \frac{Z(u)_c}{e^{Z(u)_c} - 1}(X) = \ell(e^{Z(u)_c})(X) = \ell(e^{uX_c}e^{Y_c})(X) \; , \tag{51}$$

where the series $\ell(z)$ was given in the assignment sheet. By integrating with respect to u, we get

$$Z(1) = Z(0) + \int_0^1 du\, \ell(e^{uX_c}e^{Y_c})(X) \; . \tag{52}$$

This is all quite formal, so let's just expand (41) to see what it really means. To leading order in X and Y, we have

$$e^{uX_c}e^{Y_c} - 1 \simeq uX_c + Y_c \; , \tag{53}$$

and so

$$\ell(e^{uX_c}e^{Y_c})(X) \simeq \left(1 - \frac{1}{2}(uX_c + Y_c)\right) \; . \tag{54}$$

Plugging (43) into (41), setting $Z(0) = Y$, and integrating with respect to u, we get

$$Z(1) \simeq Y + X - \frac{1}{2}\left(\frac{1}{2}X_c + Y_c\right)(X) = X + Y + \frac{1}{2}[X, Y] \;, \tag{55}$$

which is what we expected at quadratic order from part b. Given enough time, we could use (41) to calculate $Z(1)$ to any order in X and Y by making a careful Taylor expansion. We see that $Z(1)$ is equal to something that is composed of sum and commutators of X and Y, and therefore $Z(1) \in \mathcal{L}$.