Physics 129b
Problem Set Number 10
Due Wednesday, March 19, 2008

You may turn this problem set directly in to Jeffrey Atwell's mail slot.
Reading: Finish reading the note on the permutation groups and Young diagrams posted on the web.

42. Along the lines of our class discussion of the "1-D Helmholtz equation": Exercise 4 of the note on Lie Groups and Lie Algebras, linked to the course web page. [If we had some more time, we'd carry this problem further to learn about properties of the solutions, as in the example in class.]

43. From our definition of a tensor it is clear that the direct (or "outer" or "Kronecker" product of two vectors yields a second rank tensor. Can any element (second rank tensor) in the tensor space be written as the outer product of two tensors? If not, give a counter-example in the lowest dimension possible. Give a necessary condition for the outer product form to work.

44. Let
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1}$$
be a vector in a two-dimensional space. Which of the following objects, defined in terms of the components of x, are tensors?
$$\begin{pmatrix} (x_1)^2 & x_1x_2 \\ x_1x_2 & (x_2)^2 \end{pmatrix} \quad \begin{pmatrix} (x_2)^2 & x_1x_2 \\ x_1x_2 & (x_1)^2 \end{pmatrix} \quad \begin{pmatrix} x_1x_2 & (x_2)^2 \\ (x_1)^2 & -x_1x_2 \end{pmatrix} \quad \begin{pmatrix} -x_1x_2 & (x_1)^2 \\ -(x_2)^2 & x_1x_2 \end{pmatrix} \tag{2}$$
Part of this problem is to realize what the question is!

45. Some more practice with index manipulation, and getting used to summing over repeated indices would probably be useful for at least some people. Hence, prove the following familiar vector identities in three dimensions using our tensor notation. For example, write the cross product as:
$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k. \tag{3}$$
Here, ϵ_{ijk} is the Levi-Civita symbol. You will also wish to use the Kronecker symbol, δ_{ij}. [We don't need to make a distinction between raised and lowered indices here.]

(a) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$.

(b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$.

(c) $\mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] = (\mathbf{B} \cdot \mathbf{D})\mathbf{A} \times \mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} \times \mathbf{D}$.

Once you get some practice at this way of manipulating things, this is one of the simplest ways of proving such identities.