One way to see the multiplication table is to create a matrix representation on the basis $\{x, y, 1\}$. (This should remind you of an earlier problem on the Poincare group because this group is a subgroup of the extended Poincare group.)

$$\tau(\epsilon,\theta,\alpha,\beta) = \begin{pmatrix} \cos\theta & -\sin\theta & \alpha \\ \epsilon\sin\theta & \epsilon\cos\theta & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

with this representation and noticing ϵ^2 is always unity it is easy to confirm the multiplication rule given. Inverting this matrix and rearranging to match the original form:

$$\tau^{-1}(\epsilon,\theta,\alpha,\beta) = \begin{pmatrix} \cos\theta & \epsilon\sin\theta & -\alpha\cos\theta - \epsilon\beta\sin\theta \\ -\sin\theta & \epsilon\cos\theta & \alpha\sin\theta - \epsilon\beta\cos\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(-\epsilon\theta) & -\sin(-\epsilon\theta) & -\alpha\cos(-\epsilon\theta) + \beta\sin(-\epsilon\theta) \\ \epsilon\sin(-\epsilon\theta) & \epsilon\cos(-\epsilon\theta) & -\epsilon\alpha\sin(-\epsilon\theta) - \epsilon\beta\cos(-\epsilon\theta) \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tau^{-1}(\epsilon,\theta,\alpha,\beta) = \tau(\epsilon',\theta',\alpha',\beta')$$

where the last step is done to match the form of the original element τ. The new parameters can then be read off of the matrix:

$$\begin{aligned} \epsilon' &= \epsilon \\ \theta' &= -\epsilon\theta \\ \alpha' &= -\alpha\cos(-\epsilon\theta) + \beta\sin(-\epsilon\theta) \\ \beta' &= -\epsilon\alpha\sin(-\epsilon\theta) - \epsilon\beta\cos(-\epsilon\theta) \end{aligned}$$

An equally valid solution is to use the given multiplication rule and solve

$$\tau(\epsilon,\theta,\alpha,\beta)\tau(\epsilon',\theta',\alpha',\beta') = e$$

for the new primed parameters.

42 Outer product states

In general, a second-rank tensor cannot be expressed as the outer product of first-rank tensors (although all tensors can be expressed as a linear combination of outer products provided the outer products all belong to the same tensor space).

The simplest example of a nonseparable second rank tensor is the identity tensor δ_{ij}.

A second-rank tensor is separable if its determinant equals zero. To see this, consider separating a tensor c_{ij} into an outer product of vectors a_i and b_i. For the 2-dimensional case:

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

By equating the elements on each side, a non-trivial solution is only possible if $c_{11}c_{22} - c_{12}c_{21} = \det(c) = 0$. This argument can be extended to rank-2 tensors of higher dimensions by induction.

43 Tensor identification

Some preliminaries are useful to save some space later. Define a transformation $\hat{U}$ (in the group $GL(2)$) on the vector $\mathbf{x}$:

$$\hat{U}\mathbf{x} = \mathbf{x}'$$

In the basis given, this gives a matrix representation for $\hat{U}$ which I'll call U_0.

$$U_0 \mathbf{x} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix}$$

The first object (call it $\mathbf{T}_1$) is a valid tensor because it is the outer product of $\mathbf{x}$ with itself:

$$\mathbf{T}_1 = \mathbf{x}\mathbf{x}^T.$$

The transformation rule for $\mathbf{T}_1$ is

$$\hat{U}\mathbf{T}_1 = U_0\mathbf{T}_1U_0^T = (U_0\mathbf{x})(U_0\mathbf{x})^T = \mathbf{x}'\mathbf{x}'^T = \mathbf{T}_1'$$

The key to showing that an object is a 2nd-rank tensor is that the transformation requires two factors of the matrix U. In general these factors may come from different representations of the abstract group element $\hat{U}$.

The second object ($\mathbf{T}_2$) is also a tensor, although in a different representation (tensor space). Letting

$$\tilde{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}.$$

The transformation rule for this new vector is

$$\hat{U}\tilde{\mathbf{x}} = \tilde{U}\tilde{\mathbf{x}}; \qquad \tilde{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The form of $\tilde{U}$ is chosen so that group multiplication rule for two elements of the group still holds. For example:

$$\begin{aligned} \tilde{U}_1\tilde{U}_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_{0\,1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_{0\,2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_{0\,1}U_{0\,2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_{0\,12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tilde{U}_{12} \end{aligned}$$

From the definition of the new vectors it's evident that

$$\mathbf{T}_2 = \tilde{\mathbf{x}}\tilde{\mathbf{x}}^T$$

and the transformation rule is

$$\hat{U}\mathbf{T}_2 = \tilde{U}\mathbf{T}_2\tilde{U}^T = (\tilde{U}\tilde{\mathbf{x}})(\tilde{U}\tilde{\mathbf{x}})^T = \tilde{\mathbf{x}}'\tilde{\mathbf{x}}'^T$$

or in terms of the original vectors

$$\hat{U}\mathbf{T}_2 = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}\mathbf{x}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]^T$$

The fourth object ($\mathbf{T}_4$) is a tensor, although the vectors of the outer product defining it belong to two different vector spaces (or representations). Define

$$\mathbf{x}_L = \mathbf{x}; \qquad \mathbf{x}_R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}.$$

To transform these vectors:

$$\hat{U}\mathbf{x}_L = U_0\mathbf{x}_L = U_L\mathbf{x}_L; \qquad \hat{U}\mathbf{x}_R = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \mathbf{x}_R = U_R\mathbf{x}_R$$

The object $\mathbf{T}_4$ is the outer product of these two vectors

$$\mathbf{T}_4 = \mathbf{x}_L\mathbf{x}_R^T$$

The transformation rule is

$$\hat{U}\mathbf{T}_4 = U_L\mathbf{T}_4U_R^T = (U_L\mathbf{x}_L)(U_R\mathbf{x}_R)^T = \mathbf{x}_L'\mathbf{x}_R'^T = \mathbf{T}_4'$$

or in terms of the original vectors

$$\hat{U}\mathbf{T}_4 = U_0\left[\mathbf{x}\mathbf{x}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]^T$$

Returning to $\mathbf{T}_3$: first, this object cannot be written as an outer product. But in reference to Problem 42, it may still be a tensor if it obeys a transformation rule: $\hat{U}\mathbf{T}_3 = U_a \mathbf{T}_3 U_b$, where U_a and U_b are matrix representations of the operator $\hat{U}$. No such matrices exist, therefore the object $\mathbf{T}_3$ is not a tensor.

Lastly, here are some comments on common errors. Many students showed that the objects were outer products of arbitrary vectors, but failed to relate these to the original vector $\mathbf{x}$, or failed to show how the transformation rule works (which is the most important aspect of tensors). Others had incorrect forms for the matrix representations (for example, incorrect $\tilde{U}$ above) which did not preserve the group multiplication rule and were therefore invalid.

Some students failingly attempted to use the tensor package of Mathematica to solve this problem. Often this resulted in restrictions on the allowed transformations U. This is because Mathematica implicitly assumes all of these objects belong to the same tensor space (specifically $\mathbf{x} \otimes \mathbf{x}$), which isn't true. It should be noted that the restrictions given by Mathematica are equivalent to statements such as $\tilde{U} = U_0$ for $\mathbf{T}_2$, which gives a subset of operators having the same components in both representations. In the above solutions, there is no restriction on the operators. People using this incorrect method also got that $\mathbf{T}_3$ is a tensor if the operator is restricted to the identity matrix. This trivial condition is not enough to prove that an object is a tensor. The moral of the story is this: *Use your carbon-based brains, not your silicon-based ones!* (Unless, of course, you really know what Mathematica is doing, you know all of the assumption and restrictions implied by the software, and you know how to interpret the results.)

A common mistake on $\mathbf{T}_3$ was to write this as a sum of outer products. While this is ok if done correctly, some students treated x_1 and x_2 as scalars, neglecting the fact that they are components of a vector. This lead to the subtle error of adding tensors belonging to different tensor spaces, which is an invalid operation.

44 Index manipulation

First, some identities will be useful:

$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k; \qquad \mathbf{A} \cdot \mathbf{B} = \delta_{ij} A_i B_j$$

$$\delta_{ij} = \delta_{ji}; \qquad \epsilon_{ijk} = \epsilon_{jki} = -\epsilon_{kji}$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

44.1 Part a

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (\epsilon_{ijk} A_j B_k) \cdot (\epsilon_{lmn} C_m D_n) \\
&= \epsilon_{ijk} A_j B_k \epsilon_{lmn} C_m D_n \delta_{il} \\
&= (\epsilon_{ijk}\epsilon_{imn}) A_j B_k C_m D_n \\
&= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) A_j B_k C_m D_n \\
&= (\delta_{jm} A_j C_m)(\delta_{kn} B_k D_n) - (\delta_{jn} A_j D_n)(\delta_{km} B_k C_m) \\
&= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})
\end{aligned}$$

44.2 Part b

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A} \times (B_i C_j \epsilon_{ijk}) + \mathbf{B} \times (C_i A_j \epsilon_{ijk}) + \mathbf{C} \times (A_i B_j \epsilon_{ijk}) \\
&= A_l B_i C_j \epsilon_{ijk}\epsilon_{lkm} + B_l C_i A_j \epsilon_{ijk}\epsilon_{lkm} + C_l A_i B_j \epsilon_{ijk}\epsilon_{lkm} \\
&= A_l B_i C_j (\epsilon_{ijk}\epsilon_{mlk} + \epsilon_{jlk}\epsilon_{mik} + \epsilon_{lik}\epsilon_{mjk}) \\
&= A_l B_i C_j (\delta_{im}\delta_{jl} - \delta_{il}\delta_{mj} + \delta_{jm}\delta_{li} - \delta_{ji}\delta_{ml} + \delta_{lm}\delta_{ij} - \delta_{lj}\delta_{mi}) \\
&= 0
\end{aligned}$$

44.3 Part c

$$\begin{aligned}
\mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] &= \mathbf{A} \times [\mathbf{B} \times (C_i D_j \epsilon_{ijk})] \\
&= \mathbf{A} \times [B_l C_i D_j \epsilon_{ijk}\epsilon_{lkm}] \\
&= \mathbf{A} \times [B_l C_i D_j (\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm})] \\
&= \mathbf{A} \times [C_m (\mathbf{B} \cdot \mathbf{D}) - D_m (\mathbf{B} \cdot \mathbf{C})] \\
&= (\mathbf{B} \cdot \mathbf{D})(\mathbf{A} \times \mathbf{C}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \times \mathbf{D})
\end{aligned}$$