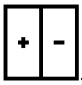
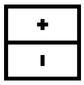
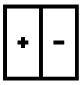
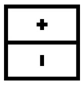
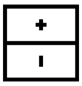
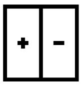
22 Equivalence of irreducible represectations

The two modes in Fig. 3.8 generate a two dimensional vector space. Let's make the following identifications:

$\begin{pmatrix}1\\0\end{pmatrix} =$ , $\quad \begin{pmatrix}0\\1\end{pmatrix} =$  (1)

Consider an element $R_{\pi/2}$ in D_4 which rotates the drum by 90 degrees counterclockwise. Its action is

 $\to (-1) \times$ 

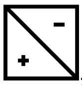
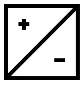
 $\to$  (2)

Hence the corresponding matrix is $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. It is straightforward to proceed in this way to get the other matrices for different group elements. The result is

e	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	M_a	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$
$R_{\pi/2}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	M_b	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
$R_{-\pi/2}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	M_α	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$
R_π	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	M_β	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

Let's call this representation D_1.

Another representation corresponding to Fig. 3.9 is constructed using the following basis:

$\begin{pmatrix}1\\0\end{pmatrix} =$ , $\quad \begin{pmatrix}0\\1\end{pmatrix} =$  (3)

Then the representation is given by

e	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	M_a	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$
$R_{\pi/2}$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	M_b	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
$R_{-\pi/2}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	M_α	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
R_π	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	M_β	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

Call this representation D_2.

At this stage, we can conclude that the two representations are equivalent since the characters of any group element in the two representations are the same. More explicitly, we can easily find a similarity transformation S that satisfies

$$SD_1(g)S^{-1} = D_2(g) \qquad \text{for all } g \in D_4. \tag{4}$$

This S can be chosen to be the rotation matrix

$$\begin{pmatrix}\cos\pi/4 & -\sin\pi/4\\ \sin\pi/4 & \cos\pi/4\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}. \tag{5}$$

The reason is that the two pairs of modes in Figs. 3.8 and 3.9 are simply related by rotation by $\pi/4$.

23 Pentagonal drumhead

23.1 Group elements and classes

The group described in the problem is the C_{5v} point group. The elements consist of a five-cycle of rotations, the mirror operation and combinations: $e, R, R^2, R^3 = R^{-2}, R^4 = R^{-1}, M, MR, MR^2, MR^3 = MR^{-2}, MR^4 = MR^{-1}$. There are four classes: identity: $\{e\}$, rotations by $2\pi/5$: $\{R, R^{-1}\}$, rotations by $4\pi/5$: $\{R^2, R^{-2}\}$, and improper rotations: $\{M, MR, MR^2, MR^3 = MR^{-2}, MR^4 = MR^{-1}\}$.

23.2 Character table

$\mathbf{C_{5v}}$	$\{e\}$ e	$\{R, R^{-1}\}$ $2C_5$	$\{R^2, R^{-2}\}$ $2C_5^2$	$\{M, \text{etc.}\}$ $5\sigma_v$
$\chi_1(A_1)$	1	1	1	1
$\chi_2(A_2)$	1	1	1	-1
$\chi_3(E_1)$	2	$2\cos(2\pi/5)$	$2\cos(4\pi/5)$	0
$\chi_4(E_2)$	2	$2\cos(4\pi/5)$	$2\cos(2\pi/5)$	0

Note $2\cos(2\pi/5) = (-1+\sqrt{5})/2$ and $2\cos(4\pi/5) = -(1+\sqrt{5})/2$.

23.3 Representation

In the representation described, there are 5 basis states corresponding to each of the 5 vertices. Representative elements from each class follow.

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad M = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad R^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Notice that the matrix for M contains a factor of -1. This is the result of a phase-shift caused by the reflections.

23.4 Decomposition into irreps

Appending the character of the new representation to the character table:

$\mathbf{C_{5v}}$	$\{e\}$ e	$\{R, R^{-1}\}$ $2C_5$	$\{R^2, R^{-2}\}$ $2C_5^2$	$\{M, \text{etc.}\}$ $5\sigma_v$
$\chi_1(A_1)$	1	1	1	1
$\chi_2(A_2)$	1	1	1	-1
$\chi_3(E_1)$	2	$2\cos(2\pi/5)$	$2\cos(4\pi/5)$	0
$\chi_4(E_2)$	2	$2\cos(4\pi/5)$	$2\cos(2\pi/5)$	0
χ_5	5	0	0	-1

The decomposition, found by inspection or by using projection operators is $\chi_5 = \chi_2 + \chi_3 + \chi_4$ or $D_5 = D_2 \oplus D_3 \oplus D_4$.

24 Atomic level splitting in cubic crystal

This problem is worth 20 points.

24.1 Show D_l is an irrep

The easiest way to show D_l is a representation is to show that the multiplication rule holds:

$$\hat{P}_R Y_{lm} = \sum_{m'=-l}^{l} Y_{lm'} D^l(R)_{m'm}$$

acting by $\hat{P}_{R'}$ on the left:

$$\hat{P}_{R'}\hat{P}_R Y_{lm} = \sum_{m''=-l}^{l} D^l(R')_{m''m'} \sum_{m'=-l}^{l} D^l(R)_{m'm} Y_{lm''}$$

The left hand side is

$$\hat{P}_{R'}\hat{P}_R Y_{lm} = \hat{P}_{R'R} Y_{lm} = \sum_{m''=-l}^{l} D^l(R'R)_{m''m} Y_{lm''}$$

The right hand side is simply a matrix product of $D^l(R')$ and $D^l(R)$, showing that this matrix product is equivalent to the product of group elements

$$D^l(R'R) = D^l(R')D^l(R).$$

Therefore, the representation obeys the group multiplication rule.

To show irreducibility, several approaches are valid. One is to show that the space spanned by the basis functions Y_{lm} has no invariant subspaces on the rotation group. Another proof is by induction, starting with D^0 as the trivial irrep and then showing that the representation D^l can't be decomposed into a direct sum of smaller irreps ($D^{l'}$ where $l' < l$).

24.2 Character of D_l

The equation

$$\hat{P}(\alpha, z)Y_{lm} = e^{-im\alpha}Y_{lm}$$

implies that Y_{lm} is an eigenfunction of $\hat{P}(\alpha, z)$ with eigenvalue $e^{-im\alpha}$. In other words, this particular element is diagonal. The character is then the sum of these eigenvalues:

$$\chi^l_\alpha = \sum_{m=-l}^{l} e^{-im\alpha}$$

shifting the index $m' = m + l$:

$$\chi^l_\alpha = \sum_{m'=0}^{2l} e^{-i(m'-l)\alpha} = e^{il\alpha}\sum_{m'=0}^{2l} e^{-im'\alpha} = e^{il\alpha}\frac{1-(e^{-i\alpha})^{2l+1}}{1-e^{-i\alpha}} = \frac{sin[(l+1/2)\alpha]}{sin[\alpha/2]}$$

24.3 Cubic group O

The group consists of 24 elements. The classes are the identity class, rotations by $\pi/2$ about a face center, rotations by π about a face center, rotations by $\pi/3$ about a vertex, and rotations by π about the line connecting the middle of opposite edges. The row under the class labels gives the angle of rotation, which is needed for the next part. The last row is the character of the rotation group representation with $l = 3$ also needed for part d.

O	e	$6C_4$	$3C_4^2$	$8C_3$	$6C_2$
angle (α)	0	$\pi/2$	π	$2\pi/3$	π
$\chi_1(A_1)$	1	1	1	1	1
$\chi_2(A_2)$	1	-1	1	1	-1
$\chi_3(E)$	2	0	2	-1	0
$\chi_4(T_1)$	3	1	-1	0	-1
$\chi_5(T_2)$	3	-1	-1	0	1
χ^3	7	-1	-1	1	-1

Note, this group is isomorphic to T_d (edges of the tetrahedron map to faces of the cube) and S_4.

24.4 Decomposition of f-level

Because the cubic group is a subgroup of the full rotation group, each of the cubic group classes is a subset of classes in the rotation group. The classes of the rotation group are labeled by the angle of rotation, so it is straightforward to calculate the characters of the cubic group classes in the $l = 3$ representation of the rotation group. Simply apply the formula from part b for each angle. For example:

$$\chi^3(C_2) = \chi^3(\pi) = \frac{\sin([3 + 1/2]\pi)}{\sin[\pi/2]} = -1$$

Care must be taken with the identity class because the quotient is indeterminate. Use L'Hôpital's Rule:

$$\chi^3(e) = \chi^3(0) = \lim_{\alpha \to 0^+} \frac{\sin[(7/2)\alpha]}{\sin[\alpha/2]} = \lim_{\alpha \to 0^+} \frac{(7/2)\cos[(7/2)\alpha]}{(1/2)\cos[\alpha/2]} = 7$$

This is the expected result because the D^3 irrep has $2(3) + 1 = 7$ dimensions.

Now that we have the character for the $l = 3$ irrep, we can decompose it into the irreps of the cubic group, either by inspection or by projection operators:

$$D^3 = D_2 \oplus D_4 \oplus D_5$$

The degeneracy of the irreps is equal to their dimension, so there are one non-degenerate level and two triply degenerate levels.

25 Lie groups exercise 1 - parameterization of $SU(n)$

A general $n \times n$ complex matrix contains $2n^2$ real parameters. The restriction to unitarity means the matrix can be written in a form $U = e^{iA}$, where $A = A^\dagger$ (A is hermitian). The hermicity of A eliminates n^2 free parameters. The "special" restriction requires $|\det U| = 1$ which removes one more degree of freedom. The total number of free parameters is then

$$2n^2 - n^2 - 1 = (n + 1)(n - 1).$$

Note that we should make sure all our constraints are independent as well. The hermiticity conditions relate distinct parameters, so they are all independent. Furthermore, hermiticity does not constrain det = 1, so that condition is independent as well.