ph129 PS 2 solutions

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Problem 5 (C. Park)

Part (a)

The Fourier transform is given by

$$f(\mathbf{y}) = \frac{1}{2\pi}\int d^2x \frac{e^{-\mu r}}{r} e^{i\mathbf{x}\cdot\mathbf{y}}$$

where $r = |y|$. In the polar coordinate system (r,θ) for the $\mathbf{x}$ vector,

$$\begin{aligned} f(\mathbf{y}) &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^\infty dr r \frac{e^{-\mu r}}{r} e^{iry\cos\theta} && (1)\\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^\infty dr e^{-\mu r + iry\cos\theta} && (2)\\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{1}{\mu - iy\cos\theta} && (3)\end{aligned}$$

Integration over r makes sense since the real part of the exponent is negative($\mu > 0$). To integrate over θ introduce a complex variable $z = e^{i\theta}$ and treat the equation above as a contour integral over the unit circle in the complex plane;

$$\begin{aligned} f(\mathbf{y}) &= \frac{1}{2\pi}\oint \frac{dz}{iz}\frac{1}{\mu - \frac{iy}{2}\left(z + \frac{1}{z}\right)} && (4)\\ &= \frac{1}{2\pi}\frac{2}{y}\oint dz \frac{1}{z^2 - \frac{2}{iy}\mu z + 1} && (5)\end{aligned}$$

Notice that the product of the two roots of the denominator in the integrand is 1. That means one of the roots lies inside the unit circle and the other outside. Explicitly,

$$z = -\frac{i\mu}{y} + i\sqrt{\frac{\mu^2}{y^2} + 1}$$

is in the unit circle. Using the residue theorem,

$$f(\mathbf{y}) = 2\pi i \frac{1}{2\pi}\frac{2}{y}\frac{1}{2i\sqrt{1+\frac{\mu^2}{y^2}}} \tag{6}$$

$$= \frac{1}{\sqrt{\mu^2+y^2}} \tag{7}$$

Part (b)
We wish to evaluate Fourier transform of the "Yukawa potential" in three dimensions:

$$\begin{aligned}
Y &= \frac{1}{(2\pi)^{3/2}}\int_{(\infty)} d^3(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{p}}\frac{e^{-\mu r}}{r}, \quad \text{where } r \equiv |\mathbf{x}| && (8)\\
&= \frac{1}{(2\pi)^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} d\phi d\cos\theta dr r^2 \frac{1}{r}\exp(-\mu r - irp\cos\theta), \quad \text{where } \mathbf{x}\cdot\mathbf{p} = rp\cos\theta\\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty\int_{-1}^1 d\cos\theta dr r e^{-\mu r} e^{-irp\cos\theta}\\
&= \frac{i}{\sqrt{2\pi}p}\int_0^\infty dr e^{-\mu r}\left(e^{-irp}-e^{irp}\right)\\
&= \frac{i}{\sqrt{2\pi}p}\left(\frac{1}{\mu+ip}-\frac{1}{\mu-ip}\right)\\
&= \sqrt{\frac{2}{\pi}}\frac{1}{\mathbf{p}^2+\mu^2}. && (9)
\end{aligned}$$

We notice in passing that the Coulomb potential corresponds to $\mu \to 0$, with

$$\int_{(\infty)} d^3(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{p}}\frac{1}{4\pi|\mathbf{x}|} = \frac{1}{\mathbf{p}^2}. \tag{10}$$

Problem 6 (S. Tulin)

The Laguerre equation

$$xf'' + (1-x)f' + \lambda f = 0 \tag{11}$$

is an equation of the form

$$\sum_{k=0}^{n}(a_k + b_k x)f^{(k)}(x) = 0 \tag{12}$$

with

$$a_2 = 0 \qquad b_2 = 1 \tag{13}$$
$$a_1 = 1 \qquad b_1 = -1 \tag{14}$$
$$a_2 = \lambda \qquad b_0 = 0\ . \tag{15}$$

Thus we have

$$u(s) = \sum_{k=0}^{n} a_k s^k = s + \lambda \tag{16}$$
$$v(s) = \sum_{k=0}^{n} b_k s^k = s(s-1)\ . \tag{17}$$

And then we get

$$F(s) = \frac{A}{v(s)} \exp\left(\int^s ds' \, \frac{u(s')}{v(s')} \right) \tag{18}$$
$$= \frac{A}{s(s-1)} \exp\left(\int^s ds' \, \frac{\lambda + s'}{s'(s'-1)} \right) \tag{19}$$
$$= \frac{A}{s} \left(\frac{s-1}{s} \right)^{\lambda}\ . \tag{20}$$

The solution to the Laguerre equation is now simply

$$f(x) = \int_C ds \, F(s) \, e^{sx} \tag{21}$$

where our contour C is any contour that satifies the following condition:

$$v(s) \, F(s) \, e^{sx} |_{C_1}^{C_2} = 0\ . \tag{22}$$

If we let $\lambda = n = 0, 1, 2...$, then there are no branch cuts in $F(s)$. There is only a pole of order $n + 1$ at $s = 0$. Thus we can choose C to be the unit circle around the origin, which trivially satifies the above condition, since $C_1 = C_2$. So now we have

$$\begin{aligned} f(x) &= 2\pi i \times Residue[F(s)e^{sx}; s=0] && (23)\\ &= \frac{2\pi i}{n!} \lim_{s\to 0} \frac{d^n}{ds^n} \left(s^n\, F(s)\, e^{sx}\right) && (24)\\ &= \frac{A}{n!} \lim_{s\to 0} \frac{d^n}{ds^n} \left((s-1)^n\, e^{sx}\right) && (25) \end{aligned}$$

where in the last line I've absorbed the factor of $2\pi i$ into the arbitrary constant A. We can write this in a slightly neater form through some algebra. First, if we let $t = s-1$, then we have

$$\begin{aligned} f(x) &= \frac{A}{n!} \lim_{t\to -1} e^x \frac{d^n}{dt^n} \left(t^n\, e^{tx}\right) && (26)\\ &= \frac{A}{n!} \lim_{t\to -1} e^x \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array}\right) \left(\left(\frac{d}{dt}\right)^m e^{tx}\right) \left(\left(\frac{d}{dt}\right)^{n-m} t^n\right) && (27)\\ &= \frac{A}{n!} \lim_{t\to -1} e^x \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array}\right) x^m e^{tx} \frac{n!}{m!} t^m && (28)\\ &= \frac{A}{n!} e^x \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array}\right) (-x)^m e^{-x} \frac{n!}{m!} && (29)\\ &= \frac{A(-1)^n}{n!} e^x \frac{d^n}{dx^n} \left(x^n\, e^{-x}\right) && (30) \end{aligned}$$

where

$$\left(\begin{array}{c} n \\ m \end{array}\right) = \frac{n!}{m!(n-m)!} \qquad (31)$$

is the binomial expansion coefficient. Isolating part of this expression,

$$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n} \left(x^n\, e^{-x}\right) \qquad (32)$$

is called Rodrigues' formula and L_n is called a Laguerre polynomial.

Problem 7

A

We want to solve the equation

$$f(x) = e^x + \lambda \int_0^2 dy\, xy\, f(y)\ . \tag{33}$$

So we have $g(x) = e^x$ and degenerate kernel

$$K(x, y) = \phi(x)\psi(y) = xy \tag{34}$$

where $\phi(x) = x$ and $\psi(y) = y$. Using the method of degenerate kernels, we want to define

$$\begin{aligned} G &= \int_0^2 dx\, \psi(x)g(x) = \int_0^2 dx\, e^x x = 1 + e^2 && (35) \\ C &= \int_0^2 dx\, \psi(x)\phi(x) = \int_0^2 dx\, x^2 = \frac{8}{3} && (36) \\ F &= \int_0^2 dx\, \psi(x)f(x)\ . && (37) \end{aligned}$$

The integral equation we want to solve is now

$$F = G + \lambda CF\ . \tag{38}$$

This gives

$$F = \frac{G}{1 - \lambda C} = \frac{1 + e^2}{1 - \frac{8}{3}\lambda} \tag{39}$$

and so

$$f(x) = g(x) + \lambda\phi(x)F = e^x + \frac{1 + e^2}{1 - \frac{8}{3}\lambda}\lambda x\ . \tag{40}$$

B

The kernel is degenerate and can be written as

$$K(x, y) = \sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y) = \sum_{i=1,2} \phi_i(x)\psi_i(y) \tag{41}$$

where

$$\psi_1(x) = \cos(x) \qquad\qquad \psi_2(x) = -\sin(x) \tag{42}$$

$$\phi_1(x) = \sin(x) \qquad\qquad \phi_2(x) = \cos(x) \tag{43}$$

Now we want to define

$$c_{ij} = \int_0^\pi dx\, \psi_i(x)\phi_j(x) = \begin{pmatrix} 0 & \pi/2 \\ -\pi/2 & 0 \end{pmatrix} . \tag{44}$$

The integral equation is now

$$f_i = \lambda c_{ij} f_j \tag{45}$$

which implies

$$f_1 = \lambda \frac{\pi}{2} f_2 \qquad\qquad f_2 = -\lambda \frac{\pi}{2} f_1. \tag{46}$$

Solving these equations, we find that we must have $\lambda = \pm 2i/\pi$. These are the eigenvalues of our homogeneous integral equation. If we plug in the eigenvalues, we find that the eigenvectors are

$$f_i = A \begin{pmatrix} \pm i \\ 1 \end{pmatrix} \tag{47}$$

where A is an arbitrary constant. Thus we have

$$f(x) = \sum_{i=1,2} u_i \phi(x) = A e^{\pm ix} \tag{48}$$

where ± in $f(x)$ corresponds to the particular eigenvalue $\lambda = \pm 2i/\pi$. If $\lambda \neq \pm 2i/\pi$, then we can have only the trivial solution $f(x) = 0$. Note that it is against the law to form linear combinations of the two solutions, e.g. if you wanted to write $f(x)$ in terms of sine and cosine. Each $f(x)$ is the solution of a different integral equation, because λ is different in each case, so it doesn't make sense to combine them.

Problem 8 (H. Chung)

Let the current flowing in the inductor be I, in the resistor side be I_R, and in the capacitor side be I_C. Then we have,

$$I(t) = I_R(t) + I_C(t) \tag{49}$$

$$V_0(t) = V(t) - L\frac{dI}{dt}(t) \tag{50}$$

$$= (I - I_C)R \tag{51}$$

$$= Q/C = \frac{1}{C}\int_0^t I_C(t')dt'. \tag{52}$$

Applying Laplace transform, we get

$$\hat{V}_0(s) = \hat{V}(s) - Ls\hat{I}(s) \tag{53}$$

$$= [\hat{I}(s) - \hat{I}_C(s)]R \tag{54}$$

$$= \frac{1}{C}\int_0^\infty \exp(-st)dt \int_0^t \hat{I}_C(t')dt' = \frac{1}{sC}\hat{I}_C(s), \tag{55}$$

where $\hat{f}(s)$ is the Laplace transform of $f(s)$.

After some calculation,

$$\hat{V}_0(s) = \hat{V}\frac{1}{s^2LC + sL/R + 1}. \tag{56}$$

Therefore,

$$V_0(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \hat{V}(s)\frac{\exp(st)}{s^2LC + sL/R + 1}ds, \tag{57}$$

where a is a real positive number, and $t > 0$.

From given situation,

$$\hat{V}(s) = V\int_0^T \exp(-st)dt = V\frac{1 - \exp(-sT)}{s} \tag{58}$$

The number of poles depends on t in Eqn. 62.
Numerator of the integral is $(1 - e^{-sT})e^{st} = e^{st} - e^{-s(T-t)}$.
When $0 < t < T$, $T - t > 0$, so the way of taking contour for the second term is different from the first term. That is, we need to take contour to the left (containing three poles, which are $s_\pm = \alpha \pm \beta$, where $\alpha = -1/2RC$ and $\beta =$

$\sqrt{(1/2RC)^2 - 1/LC}$, and $s = 0$) for the first term, and to the right (containing no pole) for the second term.
By using the residue theorem, we have

$$V_0^{(1)}(t) = \frac{V}{LC}(\frac{1}{\alpha^2 - \beta^2} + \frac{e^{\alpha t}}{2\beta}[\frac{e^{\beta t}}{\alpha + \beta} - \frac{e^{-\beta t}}{\alpha - \beta}]), \tag{59}$$

for $0 < t < T$.
When $t > T$, we take contour to the left side for all term. However, there is no pole at $s = 0$ in this case, i.e. $\frac{1}{LC} Res_{s=0}[V \frac{1-\exp(-sT)}{s} \frac{\exp(st)}{s^2 LC + sL/R + 1}] = 0$)

Using the residue theorem,

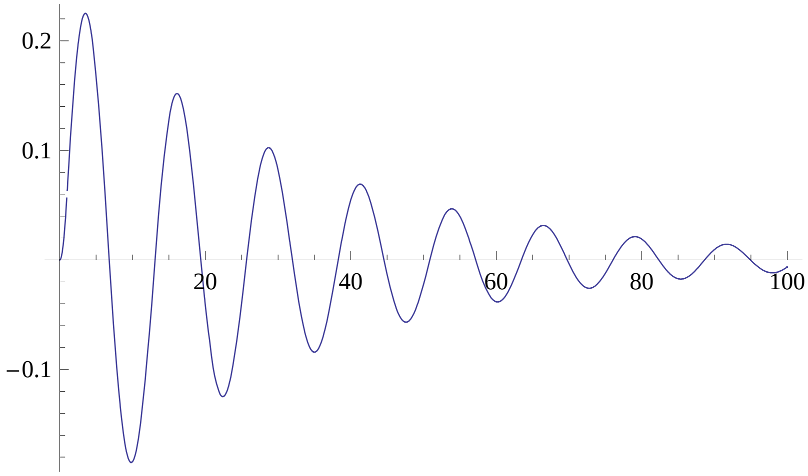
$$V_0^{(2)}(t) = \frac{1}{2\beta LC}[\hat{V}(s_+)\exp s_+ t - \hat{V}(s_-)\exp s_- t] \tag{60}$$

$$= \frac{V}{2\beta LC} e^{\alpha t}[\frac{1 - e^{-(\alpha+\beta)T}}{\alpha + \beta} e^{\beta t} - \frac{1 - e^{-(\alpha-\beta)T}}{\alpha - \beta} e^{-\beta t}], \tag{61}$$

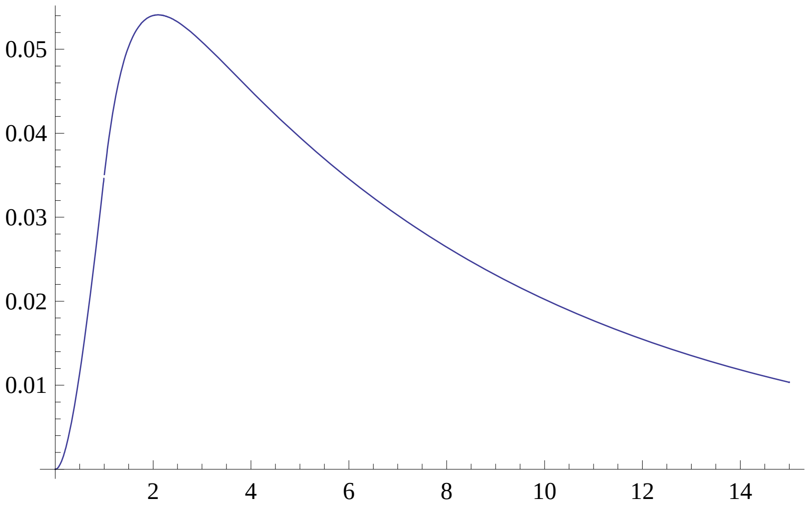
for $t > T$.

When $t = T$, we find that $V_0(T) = V_0^{(1)}(T) = V_0^{(2)}(T)$.

(a) $2RC > \sqrt{LC}$
In this case, β is imaginary. Therefore it does oscillatory motion while the amplitude gets decreased, because α is negative real number. (overdamping)



(b) $2RC < \sqrt{LC}$
In this case, β is real number. There is no oscillatory motion, and it decreases exponentially. (underdamping)



(c) $2RC = \sqrt{LC}$

In this case, β is zero. We have β in the denominator, and the equation in the bracket goes to zero as β goes to zero. We can just go back to the original equation and calculate it for the case of the pole of order 2, or we can just use L'Hopital's rule. By these ways, the answer is

$$V_0(0 < t \le T) = \frac{V}{LC}[\frac{te^{\alpha t}}{\alpha} + \frac{1 - e^{\alpha t}}{\alpha^2}] \tag{62}$$

$$V_0(t \ge T) = \frac{V}{LC}e^{\alpha t}[\frac{(\alpha T + 1)e^{-\alpha T} - 1}{\alpha^2} + \frac{1 - e^{-\alpha T}}{\alpha}t]. \tag{63}$$

So $V_0(t)$ grows before $t = T$, and decreases exponentially after $t = T$. (critical damping)