ph129 PS 3 solutions

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Problem 9

We want to solve the equation

$$f(x) = x^2 + \lambda \int_0^1 dy\, xy\, f(y)\ . \tag{1}$$

The kernel is $K(x, y) = xy$ and the inhomogenous piece is $g(x) = x^2$.

Part a: Neumann solution

The Neumann solution is an iterative solution. Let $f_n(x)$ denote $f(x)$ to order λ^n. We start with $f_0(x) = g(x) = x^2$. Then, by an iterative procedure, we can get $f_n(x)$ by

$$f_n(x) = x^2 + \lambda \int_0^1 dy\, xy f_{n-1}(y)\ . \tag{2}$$

If we use Neumann's method, we need to calculate up to $f_2(x)$ to get the terms of the second order in λ.

$$f_0(x) = g(x) = x^2 \tag{3}$$

$$f_1(x) = x^2 + \lambda \int_0^1 xyy^2 dy = x^2 + \frac{1}{4} x\lambda \tag{4}$$

$$f_2(x) = x^2 + \lambda \int_0^1 xy \left(y^2 + \frac{1}{4} y\lambda\right) dy = x^2 + \lambda x \left(\frac{1}{4} + \frac{1}{12}\lambda\right)\ . \tag{5}$$

Part b: Fredholm Method

The method goes as follows: First calculate the functions $D(\lambda)$ and $D(x, y; \lambda)$. Then the solution to the equation is simply

$$f(x) = g(x) + \int_0^1 dy\, g(y)\, \frac{D(x, y; \lambda)}{D(\lambda)}\ . \tag{6}$$

We want to work consistently to order λ^2, so we'll neglect all terms which are higher order. Then we have

$$\begin{aligned} D(x,y;\lambda) &= \lambda K(x,y) - \lambda^2 \int_0^1 dz (K(x,y)K(z,z) - K(x,z)K(z,y)) && (7) \\ &= \lambda xy - \lambda^2 \int_0^1 dz (xyz^2 - xzyz) && (8) \\ &= \lambda xy && (9) \end{aligned}$$

and

$$\begin{aligned} D(\lambda) &= 1 - \lambda \int_0^1 dx\, K(x,x) && (10) \\ &\quad + \frac{\lambda^2}{2!} \int_0^1 dx \int_0^1 dy\, (K(x,x)K(y,y) - K(x,y)K(y,x)) && (11) \\ &= 1 - \frac{\lambda}{3}\,. && (12) \end{aligned}$$

We put the pieces together and find

$$\begin{aligned} f(x) &= x^2 + \int_0^1 dy\, y\, \frac{D(x,y;\lambda)}{D(\lambda)} && (13) \\ &= x^2 + \frac{\lambda}{1 - \lambda/3} \frac{x}{4}\,. && (14) \end{aligned}$$

Note that the series for $D(\lambda)$ and $D(x,y;\lambda)$ have terminated, so we expect that this is the exact solution. In the limit of small λ, we can expand the denominator in powers of λ. Then we find

$$f(x) = x^2 + \lambda \frac{x}{4} \left(1 + \frac{\lambda}{3} + O(\lambda^2)\right), \tag{15}$$

in agreement with the terms we computed for the Neumann series solution.

c: Degenerate kernel method

The kernel is degenerate so we can easily get the answer in analytic form. Let's see what we can get from the given integral equation.

$$\begin{aligned} f(x) &= x^2 + \lambda \int_0^1 xyf(y)dy && (16) \\ &= x^2 + \lambda x \int_0^1 yf(y)dy && (17) \end{aligned}$$

If we say

$$A = \int_0^1 y f(y) dy \tag{18}$$

then $f(x) = x^2 + \lambda A x$. When we substitute this $f(x)$ into the expression for A, we get

$$A = \int_0^1 y \left(y^2 + \lambda A y\right) dy = \frac{1}{4} + \frac{1}{3}\lambda A. \tag{19}$$

This gives:

$$A = \frac{1}{4} \frac{1}{1 - \lambda/3} \tag{20}$$

If we put this value back into $f(x)$ we obtain:

$$f(x) = x^2 + \frac{\lambda}{4} \frac{1}{1 - \lambda/3} x, \tag{21}$$

in agreement with the Fredholm series solution.

Problem 10

See solution in the Integral Equations Solutions note.

Problem 11 (S. Tulin)

N=1

We want to solve this equation

$$f(x) = x + \int_0^x dy\, e^{-xy}\, f(y), \tag{22}$$

which means $K(x, y) = e^{-xy}$ and $g(x) = x$. We want to evaluate $f(1)$, and so the integral on the RHS goes from 0 to 1. The first task is to divide this integral into $N + 1$ discrete steps. For $N = 1$, we just have two values: $x_0 = 0$ and $x_1 = 1$, with $\Delta = 1$. Next, we evaluate g and K at these discrete points. If we denote $g_i = g(x_i)$ and $K_{ij} = K(x_i, x_j)$, then we have

$$g_i = (0, 1) \qquad\qquad K_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & e^{-1} \end{pmatrix}. \tag{23}$$

Now we can start with our iterative solution. First, $f_0 = g_0 = 0$. Then, the next iteration gives

$$\left(1 - \frac{1}{2}\, K_{11}\right)\, f_1 = g_1 + \left(\frac{1}{2}\, K_{10} f_0\right)\ . \tag{24}$$

Plugging in and solving for f_1, we find

$$f(1) = f_1 = \frac{1}{1 - 1/2e} = 1.225\ . \tag{25}$$

N=2

Now we have three discrete points: $x_0 = 0$, $x_1 = 1/2$, and $x_2 = 1$, with $\Delta = 1/2$. Evaluating K and g at these points, we find

$$g_i = \left(0, \frac{1}{2}, 1\right) \qquad\qquad K_{ij} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-1/4} & e^{-1/2} \\ 1 & e^{-1/2} & e^{-1} \end{pmatrix}\ . \tag{26}$$

The first iteration is the same: $f_0 = g_0 = 0$. For the next iteration we find

$$\left(1 - \frac{1}{4}\, K_{11}\right)\, f_1 = g_1 + \left(\frac{1}{4}\, K_{10} f_0\right)\ . \tag{27}$$

which gives $f_1 = 0.621$. In the final iteration, we find

$$\left(1 - \frac{1}{4}\, K_{22}\right)\, f_2 = g_2 + \frac{1}{2}\left(\frac{1}{2}\, K_{20} f_0 + K_{21} f_1\right)\ . \tag{28}$$

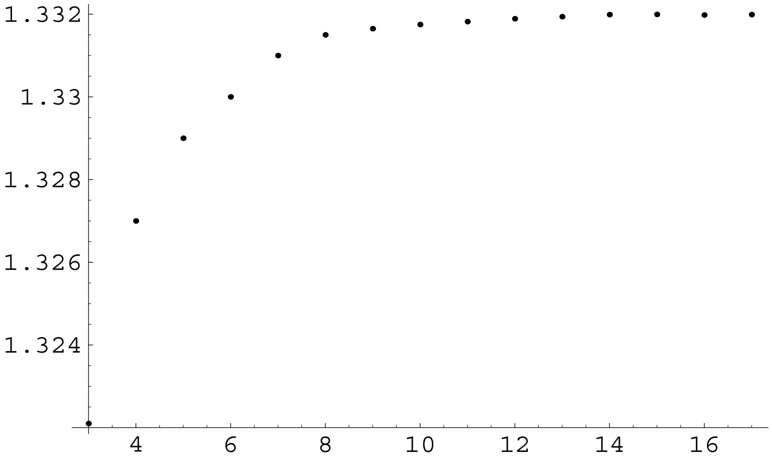
which gives $f_2 = f(1) = 1.309$.

N=3

Now $\Delta = 1/3$, so we have four discrete points: $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, and $x_3 = 1$. For K and g, we have

$$g_i = \left(0, \frac{1}{3}, \frac{2}{3}, 1\right) \qquad\qquad K_{ij} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-1/9} & e^{-2/9} & e^{-1/3} \\ 1 & e^{-2/9} & e^{-4/9} & e^{-2/3} \\ 1 & e^{-1/3} & e^{-2/3} & e{-1} \end{pmatrix}\ . \tag{29}$$

After doing the same iterative procedure as above, we find $f_1 = 0.391$, $f_2 = 0.864$, and finally $f(1) = f_3 = 1.322$. So we can see that $f(1)$ really does converge by increasing N. I've used Mathematica to repeat the procedure for larger N and the following plot shows $f(1)$ as a function of N.



Problem 12 (Integral Equations note, Exercise 20)

There are a few ways to solve the equation

$$f(x) = x + \int_0^x dy\, (y-x) f(y)\ . \tag{30}$$

A very easy way is to take derivatives. Then we get

$$\begin{aligned} f'(x) &= 1 - \int_0^x dy\, f(y) && (31) \\ f''(x) &= -f(x)\ . && (32) \end{aligned}$$

If we set $x = 0$, we have the initial condition that $f(0) = 0$. Thus, we clearly have $f(x) = \sin(x)$. Another method we can use is the Neumann method. We start with $f_0(x) = x$. Using the iterative procedure, we find that

$$\begin{aligned} f_1(x) &= x - \frac{x^3}{3!}\ , && (33) \\ f_2(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!}\ , && (34) \end{aligned}$$

and, for arbitrary n,

$$f_n(x) = \sum_{m=0}^{n} \frac{x^{2m+1}}{(2m+1)!}\ . \tag{35}$$

If we take $n \to \infty$, then $f(x) = \sin(x)$.

Problem 13 (Integral Equations note Exercise 22)

Part a:

Here we are asked to consider the Volterra equation:

$$f(x) = sin(x) + cos(x) + \int_0^x sin(x-y)f(y)dy, \tag{36}$$

and solve for $f(x)$. We take the approach of solving the equation by taking derivatives:

$$f'(x) = cos(x) - sin(x) + \int_0^x cos(x-y)f(y)dy, \tag{37}$$

$$f''(x) = -\left(sin(x) + cos(x)\right) - \int_0^x sin(x-y)f(y)dy + f(x) = 0. \tag{38}$$

We see that the second derivative vanishes by virtue of the original Volterra equation. The initial conditions are, $f(x=0) = 1$ and $f'(x=0) = 1$. Thus the solution is $f(x) = x + 1$.

Part b:

Here we are asked to consider the Volterra equation:

$$f(x) = e^{-x} + 2x + \int_0^x e^{y-x} f(y)dy, \tag{39}$$

and solve for $f(x)$. We take the approach of solving the equation by taking the derivative:

$$f'(x) = -e^{-x} + 2 + f(x) - \int_0^x e^{y-x} f(y)dy = 2x + 2. \tag{40}$$

The initial conditions is $f(x=0) = 1$, thus the solution is $f(x) = (x+1)^2$.