Ph129 PS 5 solutions

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Problem 19 (Exercise 6 In Hilbert Spaces Note)

We could let $|a_i\rangle$ be the column vector with 1 in the i'th position, and zeros in the other spots, and let $\langle b_i|$ be the i'th row of M. Then the sum yields:

$$M = \sum_{i=1}^{n} |a_i\rangle\langle b_i|.$$

Problem 20 (Exercise 7 In Hilbert Spaces Note)

(a)

$$|x| = \sum_{n=1}^{\infty} |x_n|^2 = |Cx| \ . \tag{1}$$

Hence $B = 1$ or any number greater than 1 is a good choice.

(b)

If $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ are vectors,

$$\begin{aligned} C\,(ax + by) &= C(a(x_1, x_2, \cdots) + b(y_1, y_2, \cdots)) \\ &= C(ax_1 + by_1, ax_2 + by_2, \cdots) \\ &= (0, ax_1 + by_1, ax_2 + by_2, \cdots) \\ &= a(0, x_1, x_2, \cdots) + b(0, y_1, y_2, \cdots) \\ &= aCx + bCy \end{aligned} \tag{2}$$

So, C is a linear operator.

(c)

We have to check if $\langle x|Cy\rangle = \langle Cx|y\rangle$:

$$\begin{aligned}\langle x|Cy\rangle &= \langle (x_1, x_2, \cdots)|(0, y_1, y_2, \cdots)\rangle \\ &= x_2^* y_1 + x_3^* y_2 + \cdots \\ \langle Cx|y\rangle &= \langle (0, x_1, x_2, \cdots)|(y_1, y_2, \cdots)\rangle \\ &= x_1^* y_2 + x_2^* y_3 + \cdots\end{aligned} \tag{3}$$

Therefore, $\langle x|Cy\rangle \neq \langle Cx|y\rangle$. C is not Hermitian.

(d)

$$\begin{gathered}Cx = 0 \\ (0, x_1, x_2, \cdots) = (0, 0, 0, \cdots)\end{gathered} \tag{4}$$

The equality holds component-wise. Hence $x_1 = x_2 = \cdots = 0$. There is no non-trivial solution.

(e)

In this case,

$$(0, x_1, x_2, \cdots) = (a_1, a_2, \cdots). \tag{5}$$

If $a_1 \neq 0$, there is no solution. If $a_1 = 0$, the equation is satisfied by setting

$$x_1 = a_2,\ x_2 = a_3, \cdots \tag{6}$$

If a has a finite norm, so does x. Hence there is a solution in this case.

Now, let's consider G. G is a bounded operator since $|Gx| \leq |x|$. Linearity is obvious. Note that

$$\begin{aligned}\langle x|Gy\rangle &= \langle (x_1, x_2, \cdots)|(y_1, y_2/2, \cdots)\rangle \\ &= x_1^* y_1 + x_2^* y_2/2 + \cdots \\ \langle Gx|y\rangle &= \langle (x_1, x_2/2, \cdots)|(y_1, y_2, \cdots)\rangle \\ &= x_1^* y_1 + x_2^* y_2/2 + \cdots\end{aligned} \tag{7}$$

$\langle x|Gy\rangle = \langle Cx|y\rangle$. G is Hermitian.

To have $Gx = 0$, the only possibility is $x_1 = 0, x_2 = 0, \cdots$ by matching the components. So there is only a trivial solution for this equation.

Let's check if $Gx = a$ has a solution.

$$\begin{aligned} Gx &= a \\ (x_1, x_2/2, \cdots) &= (a_1, a_2, \cdots) \\ x_1 = a_1,\ x_2 &= 2a_2, \cdots . \end{aligned} \tag{8}$$

If $a = (a_1, a_2, \cdots) = (1, 1/2, 1/3, \cdots)$, a is normalizable. But $x = (1, 1, 1, \cdots)$, which is not normalizable.

Problem 21 (Exercise 1 in Distributions Note)

See the solution in the "Solutions to selected problems" for the Distributions note.

Problem 22 (Exercise 2 in Differential Equations Note)

Let's differentiate τ once:

$$\begin{aligned} \frac{d}{dx}\tau &= e^{-\frac{a^2}{(a^2-x^2)}} \\ &= -\frac{2xa^2}{\left(a^2-x^2\right)^2} e^{-\frac{a^2}{(a^2-x^2)}} \end{aligned} \tag{9}$$

This tends to 0 as x approaches a due to the exponential factor from the left. By differentiating further, we get a more singular fraction, but still there is an exponential factor $\exp(-\frac{a^2}{(a^2-x^2)})$. Hence any higher order derivative becomes 0 near $x = a$. This means the Taylor series is just 0 around $x = a$. This can be understood by considering τ as a function of a complex variable x. Rename x by z and consider

$$\tau = e^{-\frac{a^2}{a^2-z^2}} \tag{10}$$

on the whole complex plane. When z approaches to a from the right on the real axis, τ diverges. This means that τ has a singularity at $z = a$. Since τ is defined everywhere except at $z = a$, we can make a Laurent expansion

$$\tau = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \ . \tag{11}$$

This can be done simply by expanding the exponential in τ. We get from (**??**)

$$\tau = 1 - \frac{a^2}{a^2 - z^2} + \frac{1}{2}\left(\frac{a^2}{a^2 - z^2}\right)^2 - \cdots \tag{12}$$

From this, we can expect that there will be terms with arbitrary negative powers of $z-a$ when we expand a Laurent series. Therefore, the singularity is an essential singularity. It is an isolated singularity because only one point in a complex plane is singular.

Problem 23 (Exercise 3 in Distributions Note)

See the solution in the "Solutions to selected problems" for the Distributions note.