

# Ph129 PS 6 solutions

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## Problem 24: Eigenfunctions of a kernel

We want to find the eigenvalues and eigenfunctions of the kernel:

$$k(x, y) = \frac{1}{2} \log \left| \sin \frac{x+y}{2} / \sin \frac{x-y}{2} \right| = \sum_{n=1}^{\infty} \frac{\sin nx \sin ny}{n}, \quad (1)$$

where  $0 \leq x, y \leq \pi$ .

From the definition,

$$f_m(x) = \lambda_m \int k(x, y) f_m(y) dy \quad (2)$$

$$= \lambda_m \int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin nx \sin ny}{n} f_m(y) dy \quad (3)$$

$$= \lambda_m \sum_{n=1}^{\infty} \frac{\sin nx}{n} \left( \int_0^{\pi} \sin ny f_m(y) dy \right). \quad (4)$$

From above equation, we see that eigenfunctions  $f_n(x)$  are  $\sin nx$ , and these form a complete orthonormal set over the interval  $[0, \pi]$ .

We normalize them by using identity  $\int_0^{\pi} \sin nx \sin mx dx = \frac{\pi}{2} \delta_{nm}$ :

$$f_n(x) = \sqrt{\frac{2}{\pi}} \sin nx. \quad (5)$$

Then we plug the eigenfunction into eq(4) to get the eigenvalues:

$$\sin mx = \lambda_m \sum_{n=1}^{\infty} \frac{\sin nx}{n} \int_0^{\pi} \sin ny \sin my dy \quad (6)$$

$$= \lambda_m \sum_{n=1}^{\infty} \frac{\sin nx}{n} \frac{\pi}{2} \delta_{mn} \quad (7)$$

$$= \lambda_m \frac{\pi}{2m} \sin mx \quad (8)$$

Thus, we get the eigenvalues  $\lambda_n = \frac{2n}{\pi}$ .

## Problem 25

### Problem 25: Exercise 21 in the Integral Equations course note

Our integral equation is Volterra's equation;

$$f(x) = g(x) - \lambda \int_0^x f(y) dy \quad (9)$$

We see that  $g(0) = 0$  gives  $f(0) = 0$

(a) Differentiating both sides with respect to  $x$ ,

$$f'(x) = g'(x) - \lambda f(x) \Rightarrow \left( \frac{d}{dx} + \lambda \right) f(x) = g'(x) \quad (10)$$

Thus,  $L = \frac{d}{dx} + \lambda$ .

*Linearity*

$$L(a\phi + b\psi) = \left( \frac{d}{dx} + \lambda \right) (a\phi + b\psi) \quad (11)$$

$$= \frac{d}{dx}(a\phi + b\psi) + \lambda(a\phi + b\psi) \quad (12)$$

$$= \frac{d}{dx}a\phi + \frac{d}{dx}b\psi + \lambda a\phi + \lambda b\psi \quad (13)$$

$$= a \left( \frac{d}{dx} + \lambda \right) \phi + b \left( \frac{d}{dx} + \lambda \right) \psi \quad (14)$$

$$= aL\phi + bL\psi \quad (15)$$

(b)  $G(x, y)$  is a solution of  $LG = \delta(x - y)$ , where  $\delta(x)$  is Dirac delta function.

$$Lf(x) = g'(x) = \int_0^\infty dy \delta(x - y) g'(y) \quad (16)$$

$$= \int_0^\infty dy LG(x - y) g'(y) \quad (17)$$

$$= L \int_0^\infty dy G(x - y) g'(y) \quad (18)$$

Therefore,

$$f(x) = \int_0^\infty dy G(x - y) g'(y). \quad (19)$$

In general, we can add the kernel of operator  $L$ ,  $A(x) = ae^{-\lambda x}$ , which comes from  $\left( \frac{d}{dx} + \lambda \right) A(x) = 0$ , where  $a$  is a constant. However, due to boundary condition

$A(0) = 0, a = 0$ . So eq(19) is a general solution.

(c) If  $x \neq y$ , we have  $LG(x - y) = 0$ . Then,  $G(x, y)$  is  $a(y)e^{-\lambda x}$  when  $y > x$ , and  $b(y)e^{-\lambda x}$  when  $x > y$  for some function  $a(y)$  and  $b(y)$ .

For the case of  $y > x$ , we can apply boundary condition  $G(0, y) = 0$ , and it gives  $a(y) = 0$ . So  $G(x, y) = 0$  for  $y > x$

In order to produce delta function of  $LG(x - y) = \delta(x - y)$ , we need one unit of discontinuity at  $x = y$ ,  $b(y)e^{-\lambda y} - 1 = 0$ , and it gives  $b(y) = e^{\lambda y}$ . Thus  $G(x, y) = e^{\lambda(y-x)}$  for  $x > y$

Then, we plug this information into eq(19), we get

$$f(x) = \int_0^\infty dy G(x, y) g'(y) dy = \int_0^x e^{\lambda(y-x)} g'(y) dy \quad (20)$$

## Problem 26

(Exercise 2 in Linear Differential Equations Note) See the solution in the “Solutions to selected problems” for the Differential Equations note.

## Problem 27

(Exercise 3 in Linear Differential Equations Note)

(a)

For  $\hat{p}$  to be Hermitian, we have to have  $\langle \hat{p}\phi | \psi \rangle = \langle \phi | \hat{p}\psi \rangle$ . Expanding the left side and integrating by parts we see that:

$$\langle \hat{p}\phi | \psi \rangle = -i\psi(x)\phi^*(x)|_a^b + \langle \phi | \hat{p}\psi \rangle, \quad (21)$$

so  $\hat{p}$  is Hermitian as long as  $\psi(x) \in D_{\hat{p}}$  and  $\phi(x) \in D_{\hat{p}^\dagger}$  satisfy:

$$\psi(x)\phi^*(x)|_a^b = 0. \quad (22)$$

(b)

For the momentum operator to be self-adjoint, we need to satisfy the above hermiticity condition and  $D_{\hat{p}} = D_{\hat{p}^\dagger}$  by definition of a self-adjoint operator. Thus we need to have

$$\psi(b) = \psi(a). \quad (23)$$

(c)

First we note that if the square of the operator is bounded, then the operator itself must be bounded. Since the square of the momentum operator is proportional to the kinetic energy, we see that the operator is not bounded. Consider for example an infinite square well or a harmonic oscillator. The energy in each of these systems is not bounded and thus the kinetic energy is not bounded. Therefore, the momentum operator is not bounded.

## Problem 28

(Exercise 5 in Distributions Note)

(a)

$f(t)$  is periodic with period  $T$ . We are going to find the Fourier series expansion for  $f(t)$  restricted to the interval  $[-T/2, T/2]$ . Let  $\omega$  be the Fourier transformed variable for  $t$ . The Fourier series is expanded by the exponentials  $e^{-i\omega_n t}$ . These satisfy the periodic boundary condition

$$e^{-i\omega_n t} = e^{-i\omega_n(t+T)} . \quad (24)$$

So,  $\omega_n = 2\pi n/T$ . The Fourier Series of  $f(t)$  is given by

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} s_n e^{-2\pi i n t/T} \\ s_n &= \frac{1}{T} \int_{-T/2}^{T/2} dt e^{2\pi i n t/T} f(t) \end{aligned} \quad (25)$$

Note that

$$f(t) = \begin{cases} \frac{2}{T} + \frac{4}{T^2}t & \text{if } t \in [-T/2, 0] \\ \frac{2}{T} - \frac{4}{T^2}t & \text{if } t \in [0, T/2] \end{cases} . \quad (26)$$

The integration to get  $s_n$  is straightforward and the result is

$$s_n = \frac{4}{\pi^2 n^2 T} \sin^2 \frac{\pi n}{2} = \begin{cases} 0 & \text{if } n \text{ even, except } 0 \\ \frac{1}{T} & \text{if } n = 0 \\ \frac{4}{\pi^2 n^2 T} & \text{if } n \text{ odd} \end{cases} \quad (27)$$

**(b)**

Let's treat  $d(t)$  just as an ordinary function.

$$\begin{aligned}
 \hat{d}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} d(t) \\
 &= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} \delta(t - nT) \\
 &= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\omega nT}
 \end{aligned} \tag{28}$$

$\hat{d}(\omega)$  has also an expression in terms of delta functions as shown in (38).

**(c)**

$f(t)$  is a infinite sum of some translations of  $h(t)$ :

$$f(t) = \sum_{n=-\infty}^{\infty} h(t - nT) . \tag{29}$$

This can be put into the form of convolution:

$$\begin{aligned}
 f(t) &= \int_{-\infty}^{\infty} dt' \sum_{n=-\infty}^{\infty} \delta(t' - nT) h(t - t') \\
 &= \int_{-\infty}^{\infty} dt' d(t') h(t - t')
 \end{aligned} \tag{30}$$

**(d)**

The Fourier transform  $\hat{h}(\omega)$  of  $h(t)$  is given by

$$\begin{aligned}
 \hat{h}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} h(t) \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-T/2}^0 dt e^{i\omega t} \left( \frac{2}{T} + \frac{4}{T^2} t \right) + \frac{1}{\sqrt{2\pi}} \int_0^{T/2} dt e^{i\omega t} \left( \frac{2}{T} - \frac{4}{T^2} t \right) \\
 &= \frac{1}{\sqrt{2\pi}} \frac{16}{\omega^2 T^2} \sin^2 \frac{\omega T}{4} .
 \end{aligned} \tag{31}$$

The convolution theorem is

$$\hat{f}(\omega) = \sqrt{2\pi} \hat{f}_1(\omega) \hat{f}_2(\omega) \quad \text{for } f(t) = \int_{-\infty}^{\infty} f_1(t') f_2(t-t') dt'. \quad (32)$$

The Fourier transform of the convolution (30) is then

$$\begin{aligned} \hat{f}(\omega) &= \sqrt{2\pi} \hat{d}(\omega) \hat{h}(\omega) \\ &= \frac{1}{\sqrt{2\pi}} \frac{16}{\omega^2 T^2} \sum_{n=-\infty}^{\infty} e^{i\omega n T} \sin^2 \frac{\omega T}{4}. \end{aligned} \quad (33)$$

Note that, from (27), the  $n$  th Fourier component corresponding to  $\omega = 2\pi n/T$  is

$$s_\omega = \frac{16}{\omega^2 T^3} \sin^2 \frac{\omega T}{4}. \quad (34)$$

This is the same expression as  $\hat{f}(\omega)$  except that it does not have the factor  $\hat{d}(\omega)T = \frac{T}{\sqrt{2\pi}} \sum e^{i\omega n T}$ . This factor is 0 whenever  $\omega$  is not a multiple of  $2\pi/T$ . Formally, the following identity holds:

$$\hat{d}(\omega)T = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right). \quad (35)$$

This can be seen by combining the equations in (25) with  $\omega = 2\pi n/T$ :

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt' e^{2\pi i n t' / T} f(t') e^{-2\pi i n t / T} \\ &= \frac{1}{T} \int_{-T/2}^{T/2} dt' \sum_{n=-\infty}^{\infty} e^{2\pi i n (t' - t) / T} f(t'). \end{aligned} \quad (36)$$

From this expression, we see that, formally,

$$\sum_{n=-\infty}^{\infty} e^{2\pi i n (t' - t) / T} = T \sum_{n=-\infty}^{\infty} \delta(t' - t - nT) \quad (37)$$

because  $f(t)$  is periodic with period  $T$ .

Replacing  $t' - t$  by  $\omega T^2 / 2\pi$ ,

$$\begin{aligned} \hat{d}(\omega)T &= \frac{T}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{i\omega n T} \\ &= \frac{T^2}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \delta\left(\frac{\omega T^2}{2\pi} - nT\right) \\ &= \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right). \end{aligned} \quad (38)$$