a) Replacing the weak vertex with,

\[ -i g_w \gamma^\mu \]

we get the amplitude,

\[ \mathcal{M} = \frac{g_w^2}{8(M_W)^2} \left[ \bar{u}(3) \gamma^\mu u(1) \right] \left[ \bar{u}(4) \gamma^\mu v(2) \right] \]

so that

\[ \sum |\mathcal{M}|^2 = \frac{1}{2} \left( \frac{g_w^2}{(M_W)^2} \right) [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)]. \]

This result can be found from last week’s solution by setting \( \epsilon = 0 \). For \( m_e = 0 \) and \( p_1 = (m_\mu, 0, 0, 0) \) we have the following expressions,

\[ p_1 \cdot p_2 = m_\mu E_2 \quad p_1 \cdot p_4 = m_\mu E_4 \]
\[ p_2 \cdot p_3 = 1/2(m_\mu^2 - 2m_\mu E_4) \quad p_4 \cdot p_3 = 1/2(m_\mu^2 - 2m_\mu E_2) \]

Now,

\[ \langle |\mathcal{M}|^2 \rangle = \left( \frac{g_w^2}{2(M_W)^2} \right)^2 [1/2m_\mu^2E_4(m_\mu^2 - 2E_4) + 1/2m_\mu^2E_2(m_\mu^2 - 2E_2)] \]

So with the analysis of pg 305-306 we get

\[ d\Gamma = \left( \frac{1}{8} \right)^2 \frac{g_w^4 m_\mu}{64\pi^4 M_W^4 E_4^2} \int_{1/2m_\mu - E_4}^{1/2m_\mu} [m_\mu^2E_4(m_\mu^2 - 2E_4) + m_\mu^2E_2(m_\mu^2 - 2E_2)]dE_2 \]

Integrating and setting \( E_4 = E \),

\[ \frac{d\Gamma}{dE} = \left( \frac{\pi}{2} \right) \frac{g_w^4}{(4\pi M_W)^4} m_\mu E^2 [3/2m_\mu - 8/3E] \]

This is obviously different than equation 10.35. We plot both equations below for \( 0 < E < 1/2m_\mu \). Note that in the plot energy is measured in units of \( m_\mu \) and the spectra are normalised.
We want to calculate

\[ \frac{\Gamma(K^+ \rightarrow e^- + \bar{\nu}_e)}{\Gamma(K^+ \rightarrow \mu^- + \bar{\nu}_\mu)} = \frac{m_e^2 (m_K^2 - m_e^2)^2}{m_\mu^2 (m_K^2 - m_\mu^2)^2} = 2.57 \times 10^{-5}. \]

This compares with the experimental value of \(2.44 \times 10^{-5}\). Now we are given that \(\Gamma^{-1} = 1.2 \times 10^{-8}\) s and that \(\Gamma_\mu = 0.64\Gamma\) and from,

\[ \Gamma_\mu = \frac{f_k^2}{\pi \hbar m_k^3} \left( \frac{g_W}{4M_W} \right)^4 m_\mu^2 (m_k^2 - m_\mu^2) \]

we can determine \(f_k = 36 MeV\).

If we want to calculate, say \(e^- + \nu_e \rightarrow e^- + \nu_e\), there are two Feynman diagrams that contribute to this process. It can be mediated by either a charged or a neutral current as opposed to only neutral currents in the \(\mu^- + \nu_\mu \rightarrow \mu^- + \nu_\mu\) process.

Muons are easier to experimentally produce because pions, which are copiously produced in proton colliders, preferentially decay into muons and muon neutrinos. See the previous problem.
The interaction Lagrangian is, $L_{\text{int}} = \alpha Y \bar{\psi} \psi \phi$ and the free part is $L_{\text{free}} = L - L_{\text{int}}$. Looking at the Euler-Lagrange equations for the free Lagrangian we see that $\psi$ is a spin 1/2 particle of mass $m_1$ and so has a propagator of $\frac{i}{p - m_1 c}$. Similarly $\phi$ is a spin-0 particle with mass $m_2$ and has a propagator $\frac{i}{p^2 - (m_2 c)^2}$. The vertex is drawn below and carries a factor of $i\alpha Y$. The solid, arrowed lines are the fermions and the dashed line the scalar.

![Figure 3: Vertex](image)

a) We are given a Lagrangian

$$L = \left| \left( \frac{-g}{2} \vec{\tau} \cdot \vec{W}_\mu - \frac{g'}{2} Y B_\mu \right) \phi \right|^2$$

where $\vec{\tau}_i = \vec{\sigma}_i$. Now we expand about $\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$.

We are only interested in the neutral part so,

$$L \sim \left| \left( \frac{-g}{2} \begin{pmatrix} W^3_\mu \\ 0 \\ -W^3_\mu \end{pmatrix} - \frac{g'}{2} \begin{pmatrix} B_\mu \\ 0 \\ B_\mu \end{pmatrix} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \frac{v^2}{8} \left( -gW^3_\mu + g'B_\mu \right)^2$$

b) We define,

$$\begin{pmatrix} Z^0 \\ A \end{pmatrix} = \begin{pmatrix} \cos \theta_w & -\sin \theta_w \\ \sin \theta_w & \cos \theta_w \end{pmatrix} \begin{pmatrix} W^3 \\ B \end{pmatrix}$$

where we have dropped the Lorentz indicies on $W^3, B$ etc. Looking at the Lagrangian fragment we see that we have a mass term for a neutral boson. So from

$$Z^0 = \frac{1}{\sqrt{g^2 + g'^2}} (-gW^3 + g'B)$$

with mass, $M_Z = v/2\sqrt{g^2 + g'^2}$, we have, $\cos \theta_w = -\frac{g}{\sqrt{g^2 + g'^2}}$ and $\sin \theta_w = \frac{g'}{\sqrt{g^2 + g'^2}}$, or $\theta_w = g'/g$. $A_\mu$ the boson orthogonal to $Z_0$ is massless and proportional to $(gW^3 + g'B)$. If we had kept the charged part of the Lagrangian we would have seen that there are two bosons of mass $gv/2$ hence we get that $M_W = \cos \theta_W M_Z$ as required.