Physics 125
Course Notes
Electromagnetic Interactions
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1 Introduction

We discuss in this note the interaction of electromagnetic radiation with matter. The framework remains the Schrödinger equation. However, we develop the beginnings of the notion of second quantization and field theory here.

2 Charged Particle Interaction with Electromagnetic Field

Classically, the Hamiltonian for a charged particle, with charge q, in an electromagnetic field, is

$$H = \frac{1}{2m}\left[\mathbf{p} - q\mathbf{A}(\mathbf{x},t)\right]^2 + q\Phi(\mathbf{x},t) + U(\mathbf{x},t), \tag{1}$$

where U represents any other (non-electromagnetic) potentials the particle may experience. The corresponding Schrödinger equation is:

$$i\frac{\partial\psi(\mathbf{x},t)}{\partial t} = \left\{\frac{1}{2m}\left[-i\nabla - q\mathbf{A}(\mathbf{x},t)\right]^2 + q\Phi(\mathbf{x},t) + U(\mathbf{x},t)\right\}\psi(\mathbf{x},t). \tag{2}$$

In classical electromagnetism, the physics is unaltered by a gauge transformation:

$$\mathbf{A}(\mathbf{x},t) \rightarrow \mathbf{A}'(\mathbf{x},t) = \mathbf{A}(\mathbf{x},t) + \nabla\chi(\mathbf{x},t) \tag{3}$$

$$\Phi(\mathbf{x},t) \rightarrow \Phi'(\mathbf{x},t) = \Phi(\mathbf{x},t) - \partial_t\chi(\mathbf{x},t), \tag{4}$$

where $\chi(\mathbf{x},t)$ is a scalar function of position and time. In particular, the electric and magnetic fields are unchanged under such a gauge transformation. We may investigate how the Schrödinger equation is altered under a gauge transformation:

$$i\frac{\partial\psi'(\mathbf{x},t)}{\partial t} = H'\psi'(\mathbf{x},t) \tag{5}$$

$$= \left\{ \frac{1}{2m} \left[-i\nabla - q\mathbf{A}'(\mathbf{x},t) \right]^2 + q\Phi'(\mathbf{x},t) + U(\mathbf{x},t) \right\} \psi'(\mathbf{x},t) \tag{6}$$

$$= \left\{ \frac{1}{2m} \left[-i\nabla - q\mathbf{A} - q\nabla\chi \right]^2 + q\Phi - q\partial_t\chi + U \right\} \psi' \tag{7}$$

$$= \left\{ H + \frac{q}{2m} \left[2(i\nabla^2\chi + \nabla\chi\cdot\nabla + q\mathbf{A}\cdot\nabla\chi) + q(\nabla\chi)^2 \right] - q\partial_t\chi \right\} \psi'.$$

We offer the following theorem:

Theorem: If $i\partial_t\psi = H\psi$ and $i\partial_t\psi' = H'\psi'$, then

$$\psi'(\mathbf{x},t) = e^{iq\chi(\mathbf{x},t)}\psi(\mathbf{x},t). \tag{8}$$

It is left as an exercise for the reader to prove this theorem.

Thus, under a gauge transformation, the wave function transforms according to

$$\psi(\mathbf{x},t) \rightarrow \psi'(\mathbf{x},t) = e^{iq\chi(\mathbf{x},t)}\psi(\mathbf{x},t). \tag{9}$$

Note that expectation of the momentum, $\mathbf{p} = -i\nabla$, is not gauge invariant:

$$\int d^3(\mathbf{x})\psi^*(-i\nabla)\psi \neq \int d^3(\mathbf{x})e^{-iq\chi}\psi^*(-i\nabla)e^{iq\chi}\psi \tag{10}$$

The second integral gives an extra term:

$$\int d^3(\mathbf{x})\psi^*(q\nabla\chi)\psi. \tag{11}$$

But $q\nabla\chi = q(\mathbf{A}'-\mathbf{A})$, so the quantity $\mathbf{p}-q\mathbf{A}$ *is* gauge invariant. Classically, this corresponds to the fact that this is just $m\frac{d\mathbf{x}}{dt}$. In the Heisenberg picture:

$$m\frac{d\mathbf{x}(t)}{dt} = \mathbf{p}(t) - q\mathbf{A}(\mathbf{x},t). \tag{12}$$

This corresponds to the physical observable of position. While the potentials and wave functions change under a gauge transformation, physical observables are gauge invariant quantum mechanically, as classically.

As an aside, let us remark a little further on our gauge transformation. The quantity $\chi(\mathbf{x},t)$ is an arbitrary scalar function (required to be differentiable). Thus, our electromagnetic theory is invariant under gauge transformations where the wave function transforms according to $\psi \rightarrow e^{i\chi}\psi$. The $e^{i\chi}$ factor may be regarded as a one-by-one unitary matrix. We have a group symmetry, where the transformation group is just $U(1)$, called a (local) “gauge group”. The word “local” is used because χ may vary with position. A “global” $U(1)$ symmetry corresponds to constant χ. In quantum field theory, one often starts with the symmetry (for example, the symmetry group $SU(3)$ for quantum chromodynamics) and works out the transformation properties, and hence the interactions, of the gauge fields.

Returning to electromagnetic interactions, let us rewrite the Hamiltonian in the form $H = H_0 + H_{\text{int}}$, where H_0 is the Hamiltonian in the absence of the electromagnetic fields,

$$H_0 = \frac{p^2}{2m} + V, \tag{13}$$

and hence,

$$H_{\text{int}} = -\frac{q}{2m}(\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}) + \frac{q^2}{2m}\mathbf{A}^2 + q\Phi. \tag{14}$$

If there are N particles interacting with the field, we have:

$$H_{\text{int}} = \sum_{i=1}^{N}\left\{-\frac{q_i}{2m_i}\left[\mathbf{p}_i\cdot\mathbf{A}(\mathbf{x}_i,t) + \mathbf{A}(\mathbf{x}_i,t)\cdot\mathbf{p}_i\right] + \frac{q_i^2}{2m_i}\mathbf{A}^2(\mathbf{x}_i,t) + q_i\Phi(\mathbf{x}_i,t)\right\}, \tag{15}$$

where any interactions (including electromagnetic) between the particles are included separately in V.

Suppose all of the particles have mass m and charge q. Then,

$$\sum_i q\Phi(\mathbf{x}_i,t) = \sum_i \int d^3(\mathbf{x}) q\delta^{(3)}(\mathbf{x}-\mathbf{x}_i)\Phi(\mathbf{x},t) \tag{16}$$

$$= \int d^3(\mathbf{x}) q\rho(\mathbf{x})\Phi(\mathbf{x},t), \tag{17}$$

where $\rho(\mathbf{x}) \equiv \sum_i \delta^{(3)}(\mathbf{x}-\mathbf{x}_i)$. Note that

$$\int d^3(\mathbf{x})\rho(\mathbf{x}) = N. \tag{18}$$

We interpret ρ as the number density operator. Likewise, we define a “number current density” operator according to:

$$\mathbf{j}(\mathbf{x}) \equiv \frac{1}{2m}\sum_i \left[\mathbf{p}_i\delta^{(3)}(\mathbf{x}-\mathbf{x}_i) + \delta^{(3)}(\mathbf{x}-\mathbf{x}_i)\mathbf{p}_i\right], \tag{19}$$

where we have been careful to define it such that it is a hermitian operator. Thus, we may write:

$$H_{\text{int}} = \int_{(\infty)} d^3(\mathbf{x}) \left[-q\mathbf{j}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x},t) + \frac{q^2}{2m}\rho(\mathbf{x})\mathbf{A}^2(\mathbf{x},t) + q\rho(\mathbf{x})\Phi(\mathbf{x},t) \right]. \tag{20}$$

We remark that $\mathbf{p}/m$ is not the velocity of the particle in the presence of the electromagnetic field. Instead,

$$\mathbf{v} = \frac{1}{m}\mathbf{p} - \frac{q}{m}\mathbf{A}. \tag{21}$$

Thus, the operator for the particle current density is actually

$$\mathbf{J}(\mathbf{x}) = \mathbf{j}(\mathbf{x}) - \frac{q}{m}\mathbf{A}(\mathbf{x},t)\rho(\mathbf{x}). \tag{22}$$

3 Example: Absorption of Light

We consider the absorption of light by a system of charged particles (for example, an atom). We'll make the following assumption: The fields $\mathbf{A}$ are small compared with the "atomic" fields ($\sim e/a_0^2$ for atoms) in the problem. Thus, we may neglect the $\rho\mathbf{A}^2$ term in comparison with the $\mathbf{j}\cdot\mathbf{A}$ term linear in $\mathbf{A}$.

Also, we pick a convenient gauge to work in, namely the "transverse" gauge, in which:

$$\Phi = 0, \quad \text{and} \quad \nabla\cdot\mathbf{A} = 0. \tag{23}$$

Thus, we have:

$$H_{\text{int}} = -q\int_{(\infty)} d^3(\mathbf{x})\mathbf{j}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x},t). \tag{24}$$

We'll now expand the external field in plane waves in a (large) box of volume V with periodic boundary conditions:

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mathbf{k},\boldsymbol{\epsilon}} \left(A_{\mathbf{k}\boldsymbol{\epsilon}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}}{\sqrt{V}}\boldsymbol{\epsilon} + A^*_{\mathbf{k}\boldsymbol{\epsilon}} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}}{\sqrt{V}}\boldsymbol{\epsilon}^* \right), \tag{25}$$

where $\boldsymbol{\epsilon}$ is the polarization vector, $\sum_{\boldsymbol{\epsilon}}$ sums over two orthogonal polarizations for given $\mathbf{k}$, and $\boldsymbol{\epsilon}$ is orthogonal to $\mathbf{k}$ in the transverse gauge. Where convenient, we will take the continuum limit:

$$\frac{1}{V}\sum_{\mathbf{k}} \rightarrow \int \frac{k^2 dk d\Omega}{(2\pi)^3}. \tag{26}$$

We have:

$$
\begin{aligned}
H_{\text{int}} &= -q\sum_{\mathbf{k},\boldsymbol{\epsilon}}\int_{(\infty)} d^3(\mathbf{x})\mathbf{j}(\mathbf{x})\cdot\left(A_{\mathbf{k}\boldsymbol{\epsilon}}\boldsymbol{\epsilon}\frac{e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}}{\sqrt{V}}+A^*_{\mathbf{k}\boldsymbol{\epsilon}}\boldsymbol{\epsilon}^*\frac{e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}}{\sqrt{V}}\right)\\
&= -q\sum_{\mathbf{k},\boldsymbol{\epsilon}}\left(A_{\mathbf{k}\boldsymbol{\epsilon}}\hat{\mathbf{j}}(-\mathbf{k})\cdot\boldsymbol{\epsilon}\frac{e^{-i\omega t}}{\sqrt{V}}+A^*_{\mathbf{k}\boldsymbol{\epsilon}}\hat{\mathbf{j}}(\mathbf{k})\cdot\boldsymbol{\epsilon}^*\frac{e^{i\omega t}}{\sqrt{V}}\right), \qquad (27)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\mathbf{j}}(\mathbf{k}) &= \int_{(\infty)} d^3(\mathbf{x})\mathbf{j}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \qquad (28)\\
&= \frac{1}{2m}\sum_i\left(\mathbf{p}_i e^{-i\mathbf{k}\cdot\mathbf{x}_i}+e^{-i\mathbf{k}\cdot\mathbf{x}_i}\mathbf{p}_i\right). \qquad (29)
\end{aligned}
$$

Let us calculate the absorption rate of a beam of light with this superposition of plane waves, by an atom in some state $|0\rangle$. We assume that the light is incoherent – here meaning that there aren't phase correlations among the different Fourier components. For example, the light source could be a hot gas (*e.g.*, sodium vapor), with atoms emitting independently. In this case, we can compute the result for each Fourier component separately, and then sum over the components.

Recall Fermi's golden rule:

$$
\Gamma_{0\to n} = 2\pi|\langle n|H_{\text{int}}|0\rangle|^2\delta(E_n-E_0-\omega). \qquad (30)
$$

In this case, conservation of energy includes the emitted photon, so the delta-function has the photon energy $\omega = k$ in it. Note that the atomic states may not be part of a continuum (as we assumed when we obtained the golden rule), but we will still get transition probabilities proportional to time (golden rule) as long as the incident light beam has a continuum of frequencies.

The Fourier components in our beam induce both upward and downward transitions of the atom. The upward transitions are caused by the "positive" frequency component of the perturbation, and the downward transitions by the "negative" frequency component. From the golden rule, then, the rate of upward transitions is:

$$
\Gamma_{\mathbf{k}\boldsymbol{\epsilon}}(\text{abs}; 0\to n) = 2\pi\delta(E_n-E_0-\omega)\frac{q^2}{V}|A_{\mathbf{k}\boldsymbol{\epsilon}}|^2|\langle n|\hat{\mathbf{j}}(-\mathbf{k})\cdot\boldsymbol{\epsilon}|0\rangle|^2. \qquad (31)
$$

Summing over $\mathbf{k}$, $\boldsymbol{\epsilon}$ (with two orthogonal polarizations for each $\mathbf{k}$):

$$
\Gamma(\text{abs}; 0\to n) = \frac{1}{V}\sum_{\mathbf{k},\boldsymbol{\epsilon}}2\pi\delta(E_n-E_0-\omega)q^2|A_{\mathbf{k}\boldsymbol{\epsilon}}|^2|\langle n|\hat{\mathbf{j}}(-\mathbf{k})\cdot\boldsymbol{\epsilon}|0\rangle|^2. \qquad (32)
$$

Changing the sum to an integral (with $k = \omega$) and doing the delta function integral (hence $\omega = E_n - E_0$) yields:

$$\Gamma(\text{abs}; 0 \to n) = 2\pi q^2 \frac{\omega^2}{(2\pi)^3} \int_{(4\pi)} d\Omega \sum_{\boldsymbol{\epsilon}} |A_{\mathbf{k}\boldsymbol{\epsilon}}|^2 |\langle n|\hat{\mathbf{j}}(-\mathbf{k}) \cdot \boldsymbol{\epsilon}|0\rangle|^2. \tag{33}$$

Suppose, for example, that the incident light beam subtends a solid angle $\Delta\Omega$, and is polarized, with polarization vector $\boldsymbol{\epsilon}$. According to Maxwell's equations,

$$\mathbf{E}(\mathbf{x}, t) = -\partial_t \mathbf{A}(\mathbf{x}, t) \tag{34}$$

$$\mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}(\mathbf{x}, t), \tag{35}$$

and the energy in an electromagnetic field is

$$E = \frac{1}{8\pi} \int d^3(\mathbf{x})(\mathbf{E}^2 + \mathbf{B}^2). \tag{36}$$

We thus have the energy in our superposition of plane waves, averaged over a few cycles:

$$E = \sum_{\mathbf{k},\boldsymbol{\epsilon}} \frac{\omega^2}{2\pi} |A_{\mathbf{k}\boldsymbol{\epsilon}}|^2. \tag{37}$$

With $c = 1$, this gives the rate of energy transport of our beam (in the given polarization):

$$\frac{1}{V} \sum_{\mathbf{k}} \frac{\omega^2}{2\pi} |A_{\mathbf{k}\boldsymbol{\epsilon}}|^2 = \Delta\Omega \int d\omega \frac{\omega^4}{(2\pi)^4} |A_{\mathbf{k}\boldsymbol{\epsilon}}|^2, \tag{38}$$

with units of energy per unit area per unit time. The intensity of the incident beam per unit frequency is thus

$$I(\omega) = \frac{\omega^4 |A_{\mathbf{k}\boldsymbol{\epsilon}}|^2}{(2\pi)^4} \Delta\Omega, \tag{39}$$

and we may write the absorption rate in terms of this intensity:

$$\Gamma_{\boldsymbol{\epsilon}}(\text{abs}; 0 \to n) = \frac{(2\pi)^2 q^2}{\omega^2} I(\omega) |\langle n|\hat{\mathbf{j}}(-\mathbf{k}) \cdot \boldsymbol{\epsilon}|0\rangle|^2. \tag{40}$$

The rate of downward transitions ("induced emission"), from $|n\rangle$ to $|0\rangle$ is similarly calculated to be (for polarized beam):

$$\begin{aligned} \Gamma_{\boldsymbol{\epsilon}}(\text{ind em}; n \to 0) &= \frac{1}{V} \sum_{\mathbf{k}} 2\pi\delta(E_n - E_0 - \omega) q^2 |A_{\mathbf{k}\boldsymbol{\epsilon}}|^2 |\langle 0|\hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^*|n\rangle|^2. \\ &= \frac{(2\pi)^2 q^2}{\omega^2} I(\omega) |\langle 0|\hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^*|n\rangle|^2, \end{aligned} \tag{41}$$

where $\omega = E_n - E_0$. Since

$$\langle 0|\hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^*|n\rangle = (\langle n|\hat{\mathbf{j}}(-\mathbf{k}) \cdot \boldsymbol{\epsilon}|0\rangle)^*, \tag{42}$$

we see that

$$\Gamma(\text{abs}; 0 \to n) = \Gamma(\text{ind em}; n \to 0). \tag{43}$$

Let us transform the absorption rate into a cross section. Suppose that there are $N_{\mathbf{k}\boldsymbol{\epsilon}}$ photons in the $\mathbf{k}\boldsymbol{\epsilon}$ mode of the incident beam, and let $\omega = |\mathbf{k}|$. Then the total energy in the incident beam is:

$$E = \sum_{\mathbf{k},\boldsymbol{\epsilon}} \omega N_{\mathbf{k}\boldsymbol{\epsilon}}. \tag{44}$$

This must be equal to

$$E = \sum_{\mathbf{k},\boldsymbol{\epsilon}} \frac{\omega^2}{2\pi} |A_{\mathbf{k}\boldsymbol{\epsilon}}|^2. \tag{45}$$

Hence, we have the relation

$$|A_{\mathbf{k}\boldsymbol{\epsilon}}|^2 = \frac{2\pi}{\omega} N_{\mathbf{k}\boldsymbol{\epsilon}}. \tag{46}$$

Thus, we may write the absorption and induced emission rates in terms of the number of photons:

$$\Gamma(\text{abs}; 0 \to n) = \Gamma(\text{ind em}; n \to 0) = \frac{1}{V} \sum_{\mathbf{k},\boldsymbol{\epsilon}} \frac{(2\pi)^2}{\omega} \delta(E_n - E_0 - \omega) q^2 N_{\mathbf{k}\boldsymbol{\epsilon}} |\langle n|\hat{\mathbf{j}}(-\mathbf{k}) \cdot \boldsymbol{\epsilon}|0\rangle|^2. \tag{47}$$

Now to get a total absorption cross section, we note first that the total absorption rate for a beam of $N_{\mathbf{k}\boldsymbol{\epsilon}}$ photons in mode $\mathbf{k}\boldsymbol{\epsilon}$ is:

$$\Gamma_{\mathbf{k}\boldsymbol{\epsilon}}(\text{abs}) = \frac{N_{\mathbf{k}\boldsymbol{\epsilon}}}{V} \frac{4\pi^2 q^2}{\omega} \sum_n |\langle n|\hat{\mathbf{j}}(-\mathbf{k}) \cdot \boldsymbol{\epsilon}|0\rangle|^2 \delta(E_n - E_0 - \omega). \tag{48}$$

But $\frac{N_{\mathbf{k}\boldsymbol{\epsilon}}}{V}$ is just the density of incident photons per unit volume, in the specified mode, and hence, with $c = 1$, is the incident photon flux. Thus, we define the total absorption cross section:

$$\sigma_{\mathbf{k}\boldsymbol{\epsilon}}(\text{abs}) = \frac{\Gamma_{\mathbf{k}\boldsymbol{\epsilon}}(\text{abs})}{\text{incident flux}} \tag{49}$$

$$= \frac{4\pi^2 q^2}{\omega} \sum_n |\langle n|\hat{\mathbf{j}}(-\mathbf{k}) \cdot \boldsymbol{\epsilon}|0\rangle|^2 \delta(E_n - E_0 - \omega). \tag{50}$$

4 Quantized Radiation Field

The discussion in terms of the number of photons in the beam suggests the following approach, of a *quantized radiation field*. Instead of talking about the absorption or induced emission of energy from/to a classical electromagnetic field, we can think, for example, of the absorption process as the atom making the $|0\rangle \to |n\rangle$ transition, while the electromagnetic field makes a transition from a state with "N" photons to a state with "$N-1$" photons.

Adopting this approach, we specify our incoherent beam by giving the number of photons in each ($\mathbf{k}\boldsymbol{\epsilon}$) mode. Hence, the normalized intitial state of the electromagnetic field is:

$$|N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, N_{\mathbf{k}_2\boldsymbol{\epsilon}_2}, \ldots\rangle, \tag{51}$$

where $N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}$ is the number of photons in mode $\mathbf{k}_1\boldsymbol{\epsilon}_1$, *etc.* Two states are orthogonal if any of the $N_{\mathbf{k}\boldsymbol{\epsilon}}$ differ. The absorption by an atom (or other "matter" sysytem) in state $|0\rangle$ of a photon in mode $\mathbf{k}\boldsymbol{\epsilon}$, resulting in the atom in state $|n\rangle$, corresponds to the transition between states of the entire system:

$$|0; N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, N_{\mathbf{k}_2\boldsymbol{\epsilon}_2}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}, \ldots\rangle \to |n; N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, N_{\mathbf{k}_2\boldsymbol{\epsilon}_2}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}} - 1, \ldots\rangle. \tag{52}$$

The initial energy is

$$E_i = E_0 + \sum_{\mathbf{k}',\boldsymbol{\epsilon}'} k' N_{\mathbf{k}'\boldsymbol{\epsilon}'}, \tag{53}$$

and the final energy is

$$E_f = E_n + \sum_{\mathbf{k}',\boldsymbol{\epsilon}'} k' N_{\mathbf{k}'\boldsymbol{\epsilon}'} - k. \tag{54}$$

Using the golden rule, the transition rate is:

$$\Gamma_{i\to f} = 2\pi\delta(E_n - E_0 - k)|\langle n; N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, N_{\mathbf{k}_2\boldsymbol{\epsilon}_2}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}} - 1, \ldots |H_{\text{int}}|0; N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, N_{\mathbf{k}_2\boldsymbol{\epsilon}_2}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}, \ldots\rangle|^2. \tag{55}$$

We determine the form of H_{int} in this "quantum" description by requiring that we get the same result as our treatment in terms of a classical external field. That is, we demand:

$$|\langle n; N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, N_{\mathbf{k}_2\boldsymbol{\epsilon}_2}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}} - 1, \ldots |H_{\text{int}}|0; N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, N_{\mathbf{k}_2\boldsymbol{\epsilon}_2}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}, \ldots\rangle|^2 \tag{56}$$

$$= \frac{q^2}{V}|\mathbf{A}_{\mathbf{k}\boldsymbol{\epsilon}}|^2|\langle n|\hat{\mathbf{j}}(-\mathbf{k})\cdot\boldsymbol{\epsilon}|0\rangle|^2 \tag{57}$$

$$= \frac{q^2}{V}\frac{2\pi}{\omega}N_{\mathbf{k}\boldsymbol{\epsilon}}|\langle n|\hat{\mathbf{j}}(-\mathbf{k})\cdot\boldsymbol{\epsilon}|0\rangle|^2. \tag{58}$$

Thus, $H_{\rm int}$ includes a $\hat{\mathbf{j}}(-\mathbf{k})\cdot\boldsymbol{\epsilon}$ piece acting on the matter subspace, times a piece that decreases the number of photons in the $\mathbf{k}\boldsymbol{\epsilon}$ mode by one. Referring back to our original expression for the "classical" interaction Hamiltonian, Eqn. 27, we see that our quantum version must have the corresponding form:

$$H_{\rm int} = -\frac{q}{\sqrt{V}}\sum_{\mathbf{k}\boldsymbol{\epsilon}}\left[\hat{\mathbf{j}}(-\mathbf{k})\cdot\boldsymbol{\epsilon}\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}} + \hat{\mathbf{j}}(\mathbf{k})\cdot\boldsymbol{\epsilon}^*\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}\right], \qquad (59)$$

where $\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}$ is an operator which operates on photon states, reducing the number of photons in mode $\mathbf{k}\boldsymbol{\epsilon}$ by one. The $\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}$ term is required to make the Hamiltonian Hermitian. It will shortly be seen that this term has the effect of increasing the number of photons in mode $\mathbf{k}\boldsymbol{\epsilon}$ by one.

By the orthogonality of states with different numbers of photons in any mode, we have:

$$\begin{aligned}\langle n; N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, N_{\mathbf{k}_2\boldsymbol{\epsilon}_2},\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}}-1,\ldots|H_{\rm int}|0;N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, N_{\mathbf{k}_2\boldsymbol{\epsilon}_2},\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}},\ldots\rangle \qquad (60)\\ = -\frac{q}{\sqrt{V}}\langle n|\hat{\mathbf{j}}(-\mathbf{k})\cdot\boldsymbol{\epsilon}|0\rangle\langle\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}}-1,\ldots|\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}|\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}},\ldots\rangle.\end{aligned}$$

By comparison with Eqn. 58, we have:

$$\langle\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}}-1,\ldots|\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}|\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}},\ldots\rangle = \sqrt{\frac{2\pi}{\omega}}\sqrt{N_{\mathbf{k}\boldsymbol{\epsilon}}}, \qquad (61)$$

up to a phase factor, which we choose to be one.[1]

If we take the complex conjugate of Eqn. 61, we obtain:

$$\begin{aligned}\langle\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}}-1,\ldots|\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}|\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}},\ldots\rangle^* \qquad (62)\\ = \langle\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}},\ldots|\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}|\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}}-1,\ldots\rangle\\ = \sqrt{\frac{2\pi}{\omega}}\sqrt{N_{\mathbf{k}\boldsymbol{\epsilon}}}. \qquad (63)\end{aligned}$$

That is, $\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}$ is an operator which increases the number of photons in mode $\mathbf{k}\boldsymbol{\epsilon}$ by one.

We have:

$$\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}|N_{\mathbf{k}_1\boldsymbol{\epsilon}_1},\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}},\ldots\rangle = \sqrt{\frac{2\pi}{\omega}}\sqrt{N_{\mathbf{k}\boldsymbol{\epsilon}}}|N_{\mathbf{k}_1\boldsymbol{\epsilon}_1},\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}}-1,\ldots\rangle \qquad (64)$$

$$\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}|N_{\mathbf{k}_1\boldsymbol{\epsilon}_1},\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}},\ldots\rangle = \sqrt{\frac{2\pi}{\omega}}\sqrt{N_{\mathbf{k}\boldsymbol{\epsilon}}+1}|N_{\mathbf{k}_1\boldsymbol{\epsilon}_1},\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}}+1,\ldots\rangle. \qquad (65)$$

[1]Note that we are free to choose the relative phases of states with different numbers of photons, since they are orthogonal.

Notice the close similarity with the harmonic oscillator creation and destruction operators $a^\dagger$ and a. We may regard the quantum mechanical description of the elecromagnetic radiation field as an infinite number of hamonic oscillators (one per mode), with the photon as the quntum of these oscillators.

Let us define a Hermitian "electromagnetic field operator":

$$\hat{\mathbf{A}}(\mathbf{x}) \equiv \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\boldsymbol{\epsilon}} \left(\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}} \boldsymbol{\epsilon} e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}} \boldsymbol{\epsilon}^* e^{-i\mathbf{k}\cdot\mathbf{x}} \right). \tag{66}$$

In terms of this operator, we may write the operator, $\hat{H}_{\text{int}}$, for the interaction of matter with this quantum mechanical radiation field as:

$$\hat{H}_{\text{int}} = \int d^3(\mathbf{x}) \left\{ -q\mathbf{j}(\mathbf{x}) \cdot \hat{\mathbf{A}}(\mathbf{x}) + \frac{q^2}{2m} \rho(\mathbf{x}) \left[\hat{\mathbf{A}}(\mathbf{x}) \right]^2 \right\}, \tag{67}$$

where we have now included the ρA^2 term.

To lowest order, the description of absoprtion of electromagnetic radiation in quantum mechanics is identical to the description of absorption of classical radiation – this was by construction. Let us consider the quantum description of emission. We want to determine the transition rate from a state $|n; N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}, \ldots\rangle$ with energy

$$E_i = E_n + \sum_{\mathbf{k}',\boldsymbol{\epsilon}'} k' N_{\mathbf{k}'\boldsymbol{\epsilon}'} \tag{68}$$

to a state $|0; N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}} + 1, \ldots\rangle$ with energy

$$E_f = E_0 + \sum_{\mathbf{k}',\boldsymbol{\epsilon}'} k' N_{\mathbf{k}'\boldsymbol{\epsilon}'} + k. \tag{69}$$

Using the golden rule, the desired rate (with $\omega = k$) is:

$$\Gamma_{\mathbf{k}\boldsymbol{\epsilon}}(\text{em}; n \to 0) = 2\pi\delta(E_n - E_0 - \omega) |\langle 0; \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}+1, \ldots | \hat{H}_{\text{int}} | n; \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}, \ldots\rangle|^2. \tag{70}$$

The matrix element is:

$$\begin{aligned} &\langle 0; \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}} + 1, \ldots | \hat{H}_{\text{int}} | n; \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}, \ldots\rangle \qquad (71) \\ &\quad = -\frac{q}{\sqrt{V}} \langle 0 | \hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^* | n\rangle \langle \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}} + 1, \ldots | \hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}} | \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}, \ldots\rangle \\ &\quad = -q\sqrt{\frac{2\pi}{\omega V}} \langle 0 | \hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^* | n\rangle \sqrt{N_{\mathbf{k}\boldsymbol{\epsilon}} + 1}. \qquad (72) \end{aligned}$$

Hence, the emission rate is

$$\Gamma_{\mathbf{k}\boldsymbol{\epsilon}}(\text{em}; n \to 0) = \frac{4\pi^2 q^2}{\omega V}\delta(E_n - E_0 - \omega)|\langle 0|\hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}^*|n\rangle|^2 (N_{\mathbf{k}\boldsymbol{\epsilon}} + 1). \tag{73}$$

This is not quite the same as the "induced emission" rate we calculated earlier in our classical correspondence treatment,

$$\Gamma_{\mathbf{k}\boldsymbol{\epsilon}}(\text{ind em}; n \to 0) = \frac{4\pi^2 q^2}{\omega V}\delta(E_n - E_0 - \omega)|\langle 0|\hat{\mathbf{j}}(\mathbf{k}) \cdot \boldsymbol{\epsilon}|n\rangle|^2 N_{\mathbf{k}\boldsymbol{\epsilon}}. \tag{74}$$

We now have an additional "+1" in the $N_{\mathbf{k}\boldsymbol{\epsilon}}+1$ term. That is, even if $N_{\mathbf{k}\boldsymbol{\epsilon}} = 0$, we can have the emission take place. This is "spontaneous emission", and the total emission rate is just the sum of the induced and spontaneous emission rates. Spontaneous emission may be regarded as the quantum mechanical version of classical radiation from an accelerating charge.

4.1 Vacuum Fluctuations

Because the electromagnetic field is quantized, we have "vacuum fluctuations" in the field, analogous to the "zero point" motion of a harmonic oscillator. The vacuum state of the electromagnetic field is, of course:

$$|\Omega\rangle = |0, 0, \ldots, 0, \ldots\rangle. \tag{75}$$

The expectation value of $\hat{\mathbf{A}}(\mathbf{x})$ is

$$\langle\Omega|\hat{\mathbf{A}}(\mathbf{x})|\Omega\rangle = 0. \tag{76}$$

However, the expectation value of $\hat{\mathbf{A}}(\mathbf{x})\hat{\mathbf{A}}(\mathbf{x}')$ is not zero, since it contains terms of the form

$$\begin{aligned} \langle\Omega|\hat{\mathbf{A}}_{\mathbf{k}\boldsymbol{\epsilon}}\hat{\mathbf{A}}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}|\Omega\rangle &= \langle\Omega|\hat{\mathbf{A}}_{\mathbf{k}\boldsymbol{\epsilon}}|0, 0, \ldots, 1, 0, \ldots\rangle\sqrt{\frac{2\pi}{\omega}} \\ &= \langle\Omega|\Omega\rangle\frac{2\pi}{\omega} = \frac{2\pi}{\omega} \neq 0. \end{aligned} \tag{77}$$

Hence, for example, the product of the electric fields at two different points will be non-zero. One may interpret spontaneous emission as "induced emission", due to the vacuum fluctuations of the electromagnetic field.

4.2 Einstein's A and B Coefficients

Let us formulate a statistical argument for the rate of spontaneous emission: The probability of finding a system at temperature T with energy E is given by the Boltzmann distribution, *i.e.*, is proportional to $e^{-E/T}$. Apply this to the problem of photons in a cavity with walls at temperature T. The relative probability of having N photons in mode $\mathbf{k}\boldsymbol{\epsilon}$, since $E = Nk$, is $e^{-Nk/T}$. Therefore, the average number of photons in mode $\mathbf{k}\boldsymbol{\epsilon}$ is:

$$\begin{aligned}
\langle N_{\mathbf{k}\boldsymbol{\epsilon}} \rangle &= \frac{\sum_{N=0}^{\infty} N e^{-Nk/T}}{\sum_{N=0}^{\infty} e^{-Nk/T}} && (78)\\
&= \frac{-\frac{d}{d(k/T)} \sum_{N=0}^{\infty} e^{-Nk/T}}{\sum_{N=0}^{\infty} e^{-Nk/T}} && (79)\\
&= \frac{-\frac{d}{d(k/T)} (1 - e^{-k/T})^{-1}}{(1 - e^{-k/T})^{-1}} && (80)\\
&= \frac{1}{e^{k/T} - 1}, && (81)
\end{aligned}$$

a result known as "Planck's distribution law". The average energy per mode is thus,

$$\langle E_{\mathbf{k}\boldsymbol{\epsilon}} \rangle = k \langle N_{\mathbf{k}\boldsymbol{\epsilon}} \rangle = \frac{k}{e^{k/T} - 1}. \quad (82)$$

Now, in a cavity, photons are constantly being absorbed and emitted on the walls. How must the absorption and emission rates be related in order to give the above average for $N_{\mathbf{k}\boldsymbol{\epsilon}}$? Let us consider a simplified model for the walls: Suppose that the atoms of the walls have two levels, with energies $E_0 < E_n$. According to the Boltzmann law, the probability to have an atom in the upper state is $P_n \propto e^{-E_n/T}$, and hence:

$$\frac{P_n}{P_0} = e^{-(E_n - E_0)/T}. \quad (83)$$

Consider a state of equilibrium between these atoms and radiation of frequency $\omega = E_n - E_0$ in the cavity. Photons of this energy are absorbed at a rate proportional to the number, N, of such photons present, and proportional to the probability that an atom is in its lower level, P_0:

$$\left(\frac{dN}{dt} \right)_{\text{abs}} = -BNP_0, \quad (84)$$

where the minus sign is for the absorption of photons (decrease in N), and B is a proportionality constant. Likewise, emission is induced at the rate:

$$\left(\frac{dN}{dt}\right)_{\text{ind em}} = BNP_n. \tag{85}$$

Note that the same constant of proportionality appears.

Since $P_n < P_0$, if these were the only processes involved, all of the photons would eventually be absorbed. That is, we would have an exponential decay:

$$N(t) = N(0)e^{B(P_n - P_0)t} \to 0, \quad \text{as } t \to \infty. \tag{86}$$

But there is an additional process, spontaneous emission, with a rate:

$$\left(\frac{dN}{dt}\right)_{\text{spon em}} = AP_n, \tag{87}$$

where A is another constant of proportionality. In equilibrium, $dN/dt = 0$, and $N = \langle N \rangle$. Thus,

$$-B\langle N \rangle P_0 + B\langle N \rangle P_n + AP_n = 0, \tag{88}$$

or, with $P_n/P_0 = e^{-(E_n - E_0)/T}$,

$$\langle N \rangle = \frac{A/B}{e^{(E_n - E_0)/T} - 1}. \tag{89}$$

Comparing this result with Planck's law, we find that $A = B$. Thus,

$$\left(\frac{dN}{dt}\right)_{\text{em}} = BP_n(N + 1), \tag{90}$$

in agreement with our earlier result for the quantum emission rate. We have deduced this without knowing B, simply by using "detailed balance" arguments.

5 Exercises

1. Prove the theorem:

Theorem: Let the Hamiltonian for a charged particle interacting with an electromagnetic field be H:

$$H = \frac{1}{2m}\left[\mathbf{p} - q\mathbf{A}(\mathbf{x},t)\right]^2 + q\Phi(\mathbf{x},t) + U(\mathbf{x},t), \tag{91}$$

Let H' be the Hamiltonian obtained from H by a gauge transformation:

$$\mathbf{A}(\mathbf{x},t) \rightarrow \mathbf{A}'(\mathbf{x},t) = \mathbf{A}(\mathbf{x},t) + \nabla\chi(\mathbf{x},t) \tag{92}$$

$$\Phi(\mathbf{x},t) \rightarrow \Phi'(\mathbf{x},t) = \Phi(\mathbf{x},t) - \partial_t\chi(\mathbf{x},t), \tag{93}$$

If $i\partial_t\psi = H\psi$ and $i\partial_t\psi' = H'\psi'$, then

$$\psi'(\mathbf{x},t) = e^{iq\chi(\mathbf{x},t)}\psi(\mathbf{x},t). \tag{94}$$

2. We stated that the actual number current density operator is:

$$\mathbf{J}(\mathbf{x}) = \mathbf{j}(\mathbf{x}) - \frac{q}{m}\mathbf{A}(\mathbf{x},t)\rho(\mathbf{x}), \tag{95}$$

where $\rho(\mathbf{x})$ is the number density operator,

$$\rho(\mathbf{x}) = \sum_i \delta^{(3)}(\mathbf{x} - \mathbf{x}_i), \tag{96}$$

and

$$\mathbf{j}(\mathbf{x}) = \frac{1}{2m}\sum_i \left[\mathbf{p}_i\delta^{(3)}(\mathbf{x} - \mathbf{x}_i) + \delta^{(3)}(\mathbf{x} - \mathbf{x}_i)\mathbf{p}_i\right]. \tag{97}$$

(a) Show that $\mathbf{J}(\mathbf{x})$ is a gauge invariant operator (*i.e.*, that its matrix elements are gauge invariant).

(b) Show, in the Heisenberg representation, the familiar law:

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t} + \nabla\cdot\mathbf{J}(\mathbf{x},t) = 0 \tag{98}$$

3. We defined the quantum mechanical electromagnetic field operators $\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}$ and $\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}$.

(a) Determine the commutation relations among these operators.

(b) We may define the quantum mechanical electric field operator according to:

$$\hat{\mathbf{E}}(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\boldsymbol{\epsilon}} \left(-i\omega \hat{A}_{\mathbf{k}\boldsymbol{\epsilon}} \boldsymbol{\epsilon} e^{i\mathbf{k}\cdot\mathbf{x}} + i\omega \hat{A}^{\dagger}_{\mathbf{k}\boldsymbol{\epsilon}} \boldsymbol{\epsilon}^{*} e^{-i\mathbf{k}\cdot\mathbf{x}} \right). \tag{99}$$

Make sure this definition makes sense to you. Compute the expectation value:

$$\langle \Omega | \hat{\mathbf{E}}(\mathbf{x}) \cdot \hat{\mathbf{E}}(\mathbf{x}') | \Omega \rangle \tag{100}$$

(c) Consider the average, $\hat{\bar{\mathbf{E}}}(\mathbf{x})$, of $\hat{\mathbf{E}}(\mathbf{x})$ over a small volume $\mathcal{V}$. What is

$$\langle \Omega | \left[\hat{\bar{\mathbf{E}}}(\mathbf{x}) \right]^2 | \Omega \rangle, \tag{101}$$

and what happens as $\mathcal{V} \to 0$?