Physics 125
Course Notes
Electromagnetic Interactions
Solutions to Problems
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1 Exercises

1. Prove the theorem:

Theorem: Let the Hamiltonian for a charged particle interacting with an electromagnetic field be H:

$$H = \frac{1}{2m}\left[\mathbf{p} - q\mathbf{A}(\mathbf{x},t)\right]^2 + q\Phi(\mathbf{x},t) + U(\mathbf{x},t), \tag{1}$$

Let H' be the Hamiltonian obtained from H by a gauge transformation:

$$\mathbf{A}(\mathbf{x},t) \rightarrow \mathbf{A}'(\mathbf{x},t) = \mathbf{A}(\mathbf{x},t) + \nabla\chi(\mathbf{x},t) \tag{2}$$

$$\Phi(\mathbf{x},t) \rightarrow \Phi'(\mathbf{x},t) = \Phi(\mathbf{x},t) - \partial_t\chi(\mathbf{x},t), \tag{3}$$

If $i\partial_t\psi = H\psi$ and $i\partial_t\psi' = H'\psi'$, then

$$\psi'(\mathbf{x},t) = e^{iq\chi(\mathbf{x},t)}\psi(\mathbf{x},t). \tag{4}$$

Solution: In the original gauge, the Schrödinger equation is:

$$\begin{aligned} i\frac{\partial\psi(\mathbf{x},t)}{\partial t} &= H\psi(\mathbf{x},t) \qquad (5)\\ &= \left\{\frac{1}{2m}\left[-i\nabla - q\mathbf{A}(\mathbf{x},t)\right]^2 + q\Phi(\mathbf{x},t) + U(\mathbf{x},t)\right\}\psi(\mathbf{x},t). \end{aligned}$$

In the new gauge, the Schrödinger equation is:

$$\begin{aligned} i\frac{\partial\psi'(\mathbf{x},t)}{\partial t} &= H'\psi'(\mathbf{x},t) \qquad (6)\\ &= \left\{\frac{1}{2m}\left[-i\nabla - q\mathbf{A}'(\mathbf{x},t)\right]^2 + q\Phi'(\mathbf{x},t) + U(\mathbf{x},t)\right\}\psi'(\mathbf{x},t) \end{aligned}$$

Suppose $\psi'(\mathbf{x},t) = e^{iq\chi(\mathbf{x},t)}\psi(\mathbf{x},t)$. Consider taking $e^{-iq\chi}$ times equation 7, and subtracting Eqn. 6. The left hand side of the result is:

$$\begin{aligned} e^{-iq\chi}(i\partial_t\psi') - i\partial_t\psi &= i\left[e^{-iq\chi}(e^{iq\chi}\partial_t\psi + \psi\partial_t e^{iq\chi}) - \partial_t\psi\right] \qquad (7)\\ &= -q(\partial_t\chi)\psi. \qquad (8) \end{aligned}$$

The right hand side is:

$$
\begin{aligned}
e^{-iq\chi}H'\psi' - H\psi \qquad (9)\\
&= e^{-iq\chi}\left[\frac{1}{2m}(i\nabla + q\mathbf{A} + q\nabla\chi)^2 + q\Phi - q\partial_t\chi + U\right] e^{iq\chi}\psi \\
&\quad - \left[\frac{1}{2m}(i\nabla + q\mathbf{A})^2 + q\Phi + U\right]\psi \\
&= \left[e^{-iq\chi}\frac{1}{2m}(i\nabla + q\mathbf{A} + q\nabla\chi)^2 e^{iq\chi} - \frac{1}{2m}(i\nabla + q\mathbf{A})^2\right]\psi - q(\partial_t\chi)\psi.
\end{aligned}
$$

Comparing the two sides, we find that the theorem will be verified if we can show that

$$
\left[e^{-iq\chi}(i\nabla + q\mathbf{A} + q\nabla\chi)^2 e^{iq\chi} - (i\nabla + q\mathbf{A})^2\right]\psi = 0. \qquad (10)
$$

For convenience, absorb the charge into $\mathbf{A}$ and into χ. Then we want to evaluate:

$$
\begin{aligned}
\left[e^{-i\chi}(i\nabla + \mathbf{A} + \nabla\chi)^2 e^{i\chi} - (i\nabla + \mathbf{A})^2\right]\psi \qquad (11)\\
&= \Big\{e^{-i\chi}\left[-\nabla^2 + (\mathbf{A} + \nabla\chi)^2 + i\left(2(\mathbf{A} + \nabla\chi)\cdot\nabla + \nabla\cdot\mathbf{A} + \nabla^2\chi\right)\right] e^{i\chi} \\
&\qquad + \nabla^2 - \mathbf{A}^2 - i(2\mathbf{A}\cdot\nabla + \nabla\cdot\mathbf{A})\Big\}\psi \\
&= \Big\{e^{-i\chi}\left[-\nabla^2 + 2\mathbf{A}\cdot\nabla\chi + (\nabla\chi)^2 + i\left(2(\mathbf{A} + \nabla\chi)\cdot\nabla + \nabla^2\chi\right)\right] e^{i\chi} \\
&\qquad + \nabla^2 - 2i\mathbf{A}\cdot\nabla\Big\}\psi \\
&= \Big[-i\nabla\chi\cdot(\nabla + i\nabla\chi) - i\nabla^2\chi - i\nabla\chi\cdot\nabla + 2\mathbf{A}\cdot\nabla\chi + (\nabla\chi)^2 \\
&\qquad + i\left(2i\mathbf{A}\cdot\nabla\chi + 2i(\nabla\chi)^2 + 2\nabla\chi\cdot\nabla + \nabla^2\chi\right)\Big]\psi \\
&= 0, \qquad (12)
\end{aligned}
$$

as desired.

2. We stated that the actual number current density operator is:

$$
\mathbf{J}(\mathbf{x}) = \mathbf{j}(\mathbf{x}) - \frac{q}{m}\mathbf{A}(\mathbf{x},t)\rho(\mathbf{x}), \qquad (13)
$$

where $\rho(\mathbf{x})$ is the number density operator,

$$\rho(\mathbf{x}) = \sum_i \delta^{(3)}(\mathbf{x} - \mathbf{x}_i), \tag{14}$$

and

$$\mathbf{j}(\mathbf{x}) = \frac{1}{2m} \sum_i \left[\mathbf{p}_i \delta^{(3)}(\mathbf{x} - \mathbf{x}_i) + \delta^{(3)}(\mathbf{x} - \mathbf{x}_i)\mathbf{p}_i \right]. \tag{15}$$

(a) Show that $\mathbf{J}(\mathbf{x})$ is a gauge invariant operator (*i.e.*, that its matrix elements are gauge invariant).

Solution:

$$\begin{aligned} \mathbf{J}(\mathbf{x}) &= \frac{1}{2m} \sum_i \left[\mathbf{p}_i \delta^{(3)}(\mathbf{x} - \mathbf{x}_i) + \delta^{(3)}(\mathbf{x} - \mathbf{x}_i)\mathbf{p}_i \right] - \frac{q}{m}\mathbf{A}(\mathbf{x}, t) \sum_i \delta^{(3)}(\mathbf{x} - \mathbf{x}_i), \\ &= \frac{1}{2m} \sum_i \left[(\mathbf{p}_i - q\mathbf{A}(\mathbf{x}_i))(\delta^{(3)}(\mathbf{x} - \mathbf{x}_i) + \delta^{(3)}(\mathbf{x} - \mathbf{x}_i)(\mathbf{p}_i - q\mathbf{A}(\mathbf{x}_i)) \right] \end{aligned} \tag{16}$$

Under a gauge transformation $\psi \to e^{iq\chi}\psi$ and $\mathbf{A} \to \mathbf{A} + \nabla\chi$. Also,

$$\begin{aligned} (\mathbf{p}_i - q\mathbf{A}(\mathbf{x}_i) - q\Delta\chi)(e^{iq\chi}\psi) &= e^{iq\chi}(\mathbf{p}_i - q\mathbf{A}(\mathbf{x}_i) - q\Delta\chi + q\Delta\chi)\psi \\ &= e^{iq\chi}(\mathbf{p}_i - q\mathbf{A}(\mathbf{x}_i))\psi. \end{aligned} \tag{17}$$

Thus, if $\mathbf{J} \to \mathbf{J}'$, then

$$\psi^* \mathbf{J} \psi \to \psi^* e^{-iq\chi} \mathbf{J}' e^{iq\chi} \psi \tag{18}$$

$$= \psi^* e^{-iq\chi} e^{iq\chi} \mathbf{J} \psi \tag{19}$$

$$= \psi^* \mathbf{J} \psi. \tag{20}$$

(b) Show, in the Heisenberg representation, the familiar law:

$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{x}, t) = 0 \tag{21}$$

Solution: In the Schrödinger picture, we have:

$$\rho_S(\mathbf{x}) = \sum_i \delta^{(3)}(\mathbf{x} - \mathbf{x}_i), \tag{22}$$

$$\mathbf{J}_S(\mathbf{x}) = \frac{1}{2m} \sum_i \left[(\mathbf{p}_i - q\mathbf{A}(\mathbf{x}_i))(\delta^{(3)}(\mathbf{x} - \mathbf{x}_i) + \delta^{(3)}(\mathbf{x} - \mathbf{x}_i)(\mathbf{p}_i - q\mathbf{A}(\mathbf{x}_i)) \right] \tag{23}$$

The Heisenberg operators may be obtained from the Schrödinger operators according to:

$$\rho_H(\mathbf{x},t) = e^{-iHt}\rho_S(\mathbf{x})e^{iHt}, \tag{24}$$
$$\mathbf{J}_H(\mathbf{x},t) = e^{-iHt}J_S(\mathbf{x})e^{iHt}. \tag{25}$$

The Hamiltonian is:

$$H = \frac{1}{2m}\left[-i\nabla - q\mathbf{A}(\mathbf{x},t)\right]^2 + q\Phi(\mathbf{x},t) + U(\mathbf{x},t). \tag{26}$$

In the case where $\mathbf{A}$ and ϕ are independent of time, the proof is a bit easier. Let's assume this is the case for now. Let us also work in the gauge where $\nabla \cdot \mathbf{A} = 0$.

The partial derivative of the number density operator with respect to time is:

$$\frac{\partial \rho_H}{\partial t} = e^{-iHt}\left[-iH\rho_S(\mathbf{x}) + i\rho_S(\mathbf{x})H\right]e^{iHt} \tag{27}$$
$$= -ie^{-iHt}\sum_j \left[H, \delta^{(3)}(\mathbf{x}-\mathbf{x}_j)\right]e^{iHt} \tag{28}$$
$$= -i\frac{1}{2m}e^{-iHt}\sum_j \left[\nabla^2 + 2iq\mathbf{A}(\mathbf{x})\cdot\nabla, \delta^{(3)}(\mathbf{x}-\mathbf{x}_j)\right]e^{iHt} \tag{29}$$

The remainder of the proof is to show that this is cancelled by the divergence of the current density operator. With our choice of gauge,

$$\nabla \cdot J_H(\mathbf{x},t) = e^{-iHt}\left[\nabla \cdot J_S(\mathbf{x})\right]e^{iHt}. \tag{30}$$

3. We defined the quantum mechanical electromagnetic field operators $\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}$ and $\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}$.

   (a) Determine the commutation relations among these operators.

   Solution: We start with:

$$\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}|N_{\mathbf{k}_1\boldsymbol{\epsilon}_1},\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}},\ldots\rangle = \sqrt{\frac{2\pi}{\omega}}\sqrt{N_{\mathbf{k}\boldsymbol{\epsilon}}}|N_{\mathbf{k}_1\boldsymbol{\epsilon}_1},\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}}-1,\ldots\rangle \tag{31}$$
$$\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}|N_{\mathbf{k}_1\boldsymbol{\epsilon}_1},\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}},\ldots\rangle = \sqrt{\frac{2\pi}{\omega}}\sqrt{N_{\mathbf{k}\boldsymbol{\epsilon}}+1}|N_{\mathbf{k}_1\boldsymbol{\epsilon}_1},\ldots,N_{\mathbf{k}\boldsymbol{\epsilon}}+1,\ldots\rangle \tag{32}$$

It is obvious that:

$$\left[\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}, \hat{A}_{\mathbf{k}'\boldsymbol{\epsilon}'}\right] = 0 \tag{33}$$

$$\left[\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}, \hat{A}^\dagger_{\mathbf{k}'\boldsymbol{\epsilon}'}\right] = 0. \tag{34}$$

Also,

$$\left[\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}, \hat{A}_{\mathbf{k}'\boldsymbol{\epsilon}'}\right] = 0, \text{ if } \mathbf{k} \neq \mathbf{k}' \text{ or } \boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}'. \tag{35}$$

It remains to consider:

$$\left[\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}, \hat{A}_{\mathbf{k}'\boldsymbol{\epsilon}'}\right] |N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}, \ldots\rangle = \tag{36}$$

$$\frac{2\pi}{\omega} N_{\mathbf{k}\boldsymbol{\epsilon}} |N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}, \ldots\rangle - \frac{2\pi}{\omega}(N_{\mathbf{k}\boldsymbol{\epsilon}}+1)|N_{\mathbf{k}_1\boldsymbol{\epsilon}_1}, \ldots, N_{\mathbf{k}\boldsymbol{\epsilon}}, \ldots\rangle.$$

Thus,

$$\left[\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}, \hat{A}_{\mathbf{k}'\boldsymbol{\epsilon}'}\right] = -\frac{2\pi}{\omega}. \tag{37}$$

(b) We may define the quantum mechanical electric field operator according to:

$$\hat{\mathbf{E}}(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\boldsymbol{\epsilon}} \left(-i\omega \hat{A}_{\mathbf{k}\boldsymbol{\epsilon}} \boldsymbol{\epsilon} e^{i\mathbf{k}\cdot\mathbf{x}} + i\omega \hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}} \boldsymbol{\epsilon}^* e^{-i\mathbf{k}\cdot\mathbf{x}}\right). \tag{38}$$

Make sure this definition makes sense to you. Compute the expectation value:

$$\langle\Omega|\hat{\mathbf{E}}(\mathbf{x}) \cdot \hat{\mathbf{E}}(\mathbf{x}')|\Omega\rangle \tag{39}$$

Solution:

$$\langle\Omega|\hat{\mathbf{E}}(\mathbf{x}) \cdot \hat{\mathbf{E}}(\mathbf{x}')|\Omega\rangle = \frac{1}{V}\sum_{\mathbf{k}\boldsymbol{\epsilon}}\sum_{\mathbf{k}'\boldsymbol{\epsilon}'} \omega\omega' \tag{40}$$

$$\langle\Omega| \left(-i\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}} \boldsymbol{\epsilon} e^{i\mathbf{k}\cdot\mathbf{x}} + i\hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}} \boldsymbol{\epsilon}^* e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \cdot \left(-i\hat{A}_{\mathbf{k}'\boldsymbol{\epsilon}'} \boldsymbol{\epsilon}' e^{i\mathbf{k}'\cdot\mathbf{x}'} + i\hat{A}^\dagger_{\mathbf{k}'\boldsymbol{\epsilon}'} \boldsymbol{\epsilon}'^* e^{-i\mathbf{k}'\cdot\mathbf{x}'}\right) |\Omega\rangle$$

$$= \frac{1}{V}\sum_{\mathbf{k}\boldsymbol{\epsilon}} \omega^2 \frac{2\pi}{\omega} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tag{41}$$

$$= 2\int_{(\infty)} \frac{d^3\mathbf{k}}{(2\pi)^3} 2\pi\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tag{42}$$

We have:

$$\langle\Omega|\hat{\mathbf{E}}(\mathbf{x}) \cdot \hat{\mathbf{E}}(\mathbf{x}')|\Omega\rangle = \frac{4\pi}{V}\sum_{\mathbf{k}} \omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \frac{1}{2\pi^2}\int_{(\infty)} d^3\mathbf{k}\,\omega e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{43}$$

Let's work further with the integral form. Let $\mathbf{r} \equiv \mathbf{x} - \mathbf{x}'$, $r = |\mathbf{r}|$, and $\omega = k = |\mathbf{k}|$. Let θ be the angle between $\mathbf{k}$ and $\mathbf{r}$. Then,

$$\begin{aligned}
\langle\Omega|\hat{\mathbf{E}}(\mathbf{x}) \cdot \hat{\mathbf{E}}(\mathbf{x}')|\Omega\rangle &= \frac{1}{\pi}\int_0^\infty k^3 dk \int_{-1}^{1} d\cos\theta e^{ikr\cos\theta} && (44)\\
&= \frac{2}{\pi r}\int_0^\infty k^2 \sin krdk && (45)\\
&= \frac{2}{\pi r^4}\int_0^\infty x^2 \sin xdx. && (46)
\end{aligned}$$

(c) Consider the average, $\hat{\bar{\mathbf{E}}}(\mathbf{x})$, of $\hat{\mathbf{E}}(\mathbf{x})$ over a small volume $\mathcal{V}$. What is

$$\langle\Omega|\left[\hat{\bar{\mathbf{E}}}(\mathbf{x})\right]^2|\Omega\rangle, \qquad (47)$$

and what happens as $\mathcal{V} \to 0$?

Solution: First, find the average of of $\hat{\mathbf{E}}(\mathbf{x})$ over a small volume $\mathcal{V}$. Suppose the volume is cubic (it might actually be more convenient to use a spherical volume), with $\mathcal{V} = \ell^3$. We need:

$$\begin{aligned}
\frac{1}{\mathcal{V}}\int_{\mathcal{V}} e^{\pm i\mathbf{k}\cdot\mathbf{x}} d^3(\mathbf{x}) &= \prod_{j=1}^{3}\frac{1}{\ell}\int_{-\ell/2}^{\ell/2} e^{\pm ik_j x_j} dx_j && (48)\\
&= \prod_{j=1}^{3}\frac{1}{\ell}\frac{1}{ik_j}2i\sin k_j\ell/2 && (49)\\
&= \prod_{j=1}^{3}\frac{2}{k_j\ell}\sin k_j\ell/2 && (50)\\
&\approx 1 \quad \text{for } k_j\ell \text{ small.} && (51)
\end{aligned}$$

Hence,

$$\begin{aligned}
\hat{\bar{\mathbf{E}}}(\mathbf{x}) &= \frac{1}{\mathcal{V}}\int_{\mathcal{V}} \hat{\mathbf{E}}(\mathbf{x}) d^3(\mathbf{x}) && (52)\\
&= \frac{1}{\sqrt{V}}\sum_{\mathbf{k}\boldsymbol{\epsilon}} i\omega\left[-\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}\boldsymbol{\epsilon}\frac{1}{\mathcal{V}}\int_{\mathcal{V}} e^{i\mathbf{k}\cdot\mathbf{x}} d^3(\mathbf{x}) + \hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}\boldsymbol{\epsilon}^*\frac{1}{\mathcal{V}}\int_{\mathcal{V}} e^{-i\mathbf{k}\cdot\mathbf{x}} d^3(\mathbf{x})\right]\\
&= \frac{1}{\sqrt{V}}\sum_{\mathbf{k}\boldsymbol{\epsilon}} i\omega\left[-\hat{A}_{\mathbf{k}\boldsymbol{\epsilon}}\boldsymbol{\epsilon} + \hat{A}^\dagger_{\mathbf{k}\boldsymbol{\epsilon}}\boldsymbol{\epsilon}^*\right]\prod_{j=1}^{3}\frac{2}{k_j\ell}\sin k_j\ell/2. && (53)
\end{aligned}$$

The vacuum expectation of the square of this averaged operator is:

$$\begin{aligned} \langle\Omega| \left[\hat{\bar{\mathbf{E}}}(\mathbf{x})\right]^2 |\Omega\rangle &= \frac{1}{V}\sum_{\mathbf{k}\boldsymbol{\epsilon}}(-\omega^2)\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}^*\left(-\frac{2\pi}{\omega}\right)\prod_{j=1}^{3}\left(\frac{2}{k_j\ell}\right)^2 \sin^2 k_j\ell/2 \\ &= \frac{4\pi}{V}\sum_{\mathbf{k}}\omega\prod_{j=1}^{3}\left(\frac{2}{k_j\ell}\right)^2 \sin^2 k_j\ell/2. \end{aligned} \tag{54}$$

Thus, for $k \ll 1/\ell$ we sum the photon energies, and for $k \gg \ell$ the terms in the sum fall off rapidly with increasing k. As the volume $\mathcal{V}$ becomes smaller and smaller, the sum gets larger and larger, with the "limit":

$$\langle\Omega| \left[\hat{\bar{\mathbf{E}}}(\mathbf{x})\right]^2 |\Omega\rangle \to \frac{4\pi}{V}\sum_{\mathbf{k}}\omega. \tag{55}$$

Note that the energy density in an electromagnetic field $\mathbf{E}$ is $\mathbf{E}^2/8\pi$. Thus, our result corresponds to an energy density of

$$\frac{1}{V}\sum_{\mathbf{k}}\frac{\omega}{2}. \tag{56}$$

This gives a total energy divergent as k^4. The result is readily interpreted in terms of the harmonic oscillator picture of our second quantization: We have an infinite number of harmonic oscillators corresponding to the different modes of the photon field. Each oscillator has a zero point energy of $\omega/2$ where $\omega = |\mathbf{k}|$ for that oscillator. Summing over the zero point energies gives an infinite result.