**Physics 195a**
**Course Notes**
**The $K^0$: An Interesting Example of a "Two-State" System**
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# 1 Introduction

An example of a two-state system is considered. Interesting complexities involve:

1. Treatment of a decaying particle.
2. Superposition of states with different masses.

# 2 The $K^0$ Meson and its Anti-particle

The $K^0$ meson is a pseudoscalar state consisting (for present purposes) of a $d$ quark and an $\bar{s}$ antiquark:

$$|K^0\rangle = |d\bar{s}\rangle. \tag{1}$$

Its antiparticle is the $\overline{K}^0$:

$$|\overline{K}^0\rangle = |\bar{d}s\rangle. \tag{2}$$

If we define a "strangeness" operator, $S$, (which counts strange quarks), these states are eigenstates, with:

$$S|K^0\rangle = |K^0\rangle, \tag{3}$$
$$S|\overline{K}^0\rangle = -|\overline{K}^0\rangle. \tag{4}$$

We may write $S$ as the two-by-two matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the $|K^0\rangle$, $|\overline{K}^0\rangle$ basis, but a convenient "basis-independent" form is:

$$S = |K^0\rangle\langle K^0| - |\overline{K}^0\rangle\langle\overline{K}^0|. \tag{5}$$

These are not eigenstates of $C$, the charge conjugation operator (which changes particles to antiparticles). It is convenient to pick the antiparticle phases such that:

$$C|K^0\rangle = -|\overline{K}^0\rangle, \tag{6}$$
$$C|\overline{K}^0\rangle = -|K^0\rangle. \tag{7}$$

If we multiply the $C$ operator by the parity operator $P$, we have:

$$CP|K^0\rangle = |\overline{K}^0\rangle, \tag{8}$$
$$CP|\overline{K}^0\rangle = |K^0\rangle. \tag{9}$$

We thus have, in the $|K^0\rangle$, $|\overline{K}^0\rangle$ basis:

$$CP = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{10}$$

which we may also express in the basis-independent form:

$$CP = |K^0\rangle\langle\overline{K}^0| + |\overline{K}^0\rangle\langle K^0|. \tag{11}$$

The eigenstates of $CP$ are (the choice of nomenclature will shortly be motivated):

$$|K_S^0\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\overline{K}^0\rangle\right), \quad \text{with } CP = +1, \tag{12}$$
$$|K_L^0\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\overline{K}^0\rangle\right), \quad \text{with } CP = -1. \tag{13}$$

We remark that the $K^0$ (or $\overline{K}^0$) is the lowest mass particle containing the strange quark. Thus, the only permitted decays must be via the weak interaction. To a good approximation (but not exactly!), $CP$ is conserved in the weak interaction (and even more so in the strong and electromagnetic interactions); we shall assume this here.

A neutral $K$ meson ($K^0$ or $\overline{K}^0$) is observed to decay sometimes to two pions and sometimes to three pions. For example, consider the observed process $K^0 \to \pi^0\pi^0$. Since all of the particles in this decay are spinless, the decay must proceed with zero orbital angular momentum ("$S$-wave" decay). Thus, the parity of the $\pi^0\pi^0$ system in the final state must be positive. But we said that the $K^0$ is a pseudoscalar particle, *i.e.*, has negative parity. Thus, this is a parity-violating decay. The weak interaction is known to violate parity (*i.e.*, parity is not conserved in the weak interaction), so this is all right. The $\pi^0$ is its own anti-particle, hence the $|\pi^0\pi^0\rangle$ final state is an eigenstate of $C$ with eigenvalue $+1$. Thus, the $|\pi^0\pi^0\rangle$ final state is also an eigenstate of $CP$ with eigenvalue $+1$.

Under our approximation that $CP$ is conserved in the weak interaction, we therefore conclude that the observation of a $K^0 \to \pi^0\pi^0$ decay projects out the $K_S^0$ component of the $K^0$ meson (likewise for the $\overline{K}^0$). The $2\pi$ decay mode is favored by phase space over decays to greater numbers of pions. However, the $K_L^0 \to 2\pi$ decay is forbidden by $CP$ conservation. Hence, the $K_L^0 \to 3\pi$ decay is important for $K_L^0$. Because the phase space is considerably suppressed, the $K_L^0$ decay rate is much slower than the $K_S^0$ rate. The observed lifetimes of the $K_S^0$ and $K_L^0$ are, respectively:

$$\tau_S = 9 \times 10^{-11} \text{ s} \quad (90 \text{ ps}), \tag{14}$$
$$\tau_L = 5 \times 10^{-8} \text{ s} \quad (50 \text{ ns}). \tag{15}$$

# 3 Time Evolution of a Kaon State

Suppose that at time $t = 0$ we have the state

$$\psi(0) = |K_S^0\rangle. \tag{16}$$

How does this state evolve in time? We should have, at time $t$,

$$\psi(t) = e^{-iHt}|K_S^0\rangle. \tag{17}$$

For a free particle, the energy is $\omega_S = \sqrt{p^2 + m_S^2}$, where $m_S$ is the mass of the $K_S^0$. But if we just use this for $H$, we won't have a particle which decays in time. We know that, if we start with a particle at $t = 0$ the probability to find it undecayed at a later time $t$ if it has a lifetime $\tau_S = 1/\Gamma_S$ is:

$$P(t) = e^{-\Gamma_S t}. \tag{18}$$

Thus, the amplitude should have an $\exp(-\Gamma_S t/2)$ time dependence, in addition to the phase variation:

$$\psi(t) = e^{-i\omega_S t - \Gamma_S t/2}|K_S^0\rangle. \tag{19}$$

Letting $\omega_L = \sqrt{p^2 + m_L^2}$, where $m_L$ is the mass of the $K_L^0$, and $\Gamma_L = 1/\tau_L$, we similarly have for an intial $K_L^0$ state ($\psi(0) = |K_L^0\rangle$):

$$\psi(t) = e^{-i\omega_L t - \Gamma_L t/2}|K_L^0\rangle. \tag{20}$$

In the $|K_S^0\rangle, |K_L^0\rangle$ basis, the Hamiltonian operator is:

$$H = \begin{pmatrix} \omega_S - i\Gamma_S/2 & 0 \\ 0 & \omega_L - i\Gamma_L/2 \end{pmatrix}. \tag{21}$$

This requires some further discussion. For example, how did I know that $H$ is diagonal in this basis (and not, perhaps, in the $|K^0\rangle, |\overline{K}^0\rangle$ basis)? The answer is that we are assuming that $CP$ is conserved. Hence, $[H, CP] = 0$. The Hamiltonian cannot mix states of differing $CP$ quantum numbers, so there are no off-diagonal terms in $H$ in the $|K_S^0\rangle, |K_L^0\rangle$ basis. The second point is that we have allowed the possibility that the masses of the two $CP$ eigenstates are not the same (having already noted that the lifetimes are different). This might be a bit worrisome – the $C$ operation does not change mass.[1] However, the $|K_S^0\rangle$ and $|\overline{K}_L^0\rangle$ are not antiparticles of one another, so there is no constraint that their masses must be equal. So, we allow the possibility that they may be different. We will address shortly the measurement of the mass difference.

Now suppose that at time $t = 0$ we have a pure $\overline{K}^0$ state:

$$\psi(0) = |\overline{K}^0\rangle. \tag{22}$$

Experimentally, this is a reasonable proposition, since we may produce such states via the strong interaction. For example, if we collide two particles with no initial strangeness (perhaps a proton and an anit-proton), we make strange particles in "associated production", *i.e.*, in the production of $s\bar{s}$ pairs. Thus, we might have the reaction $\bar{p}p \to n\overline{\Lambda K}^0$ (see Fig. 1). The presence of the $\overline{\Lambda}$, which contains the $\bar{s}$ quark, tells us that the kaon produced is a $\overline{K}^0$, since it contains the $s$ quark.

So, we can realistically imagine producing a $\overline{K}^0$ at $t = 0$. But the time-evolution to later times is governed by the Hamiltonian, which is not diagonal in the $|K^0\rangle, |\overline{K}^0\rangle$ basis. Thus, we might expect that at some later time we may observe a $K^0$. What is the probability, $P_{K^0}(t)$ that a $K^0$ meson is observed at time $t$, given a pure $\overline{K}^0$ state at $t = 0$? The answer, noting that $\psi(0) = |\overline{K}^0\rangle = \left(|K_S^0\rangle - |K_L^0\rangle\right)/\sqrt{2}$, is:

$$\begin{aligned} P_{K^0}(t) &= |\langle K^0|\psi(t)\rangle|^2 & (23)\\ &= \frac{1}{2}|\langle K^0|K_S^0\rangle e^{-i\omega_S t - \Gamma_S t/2} - \langle K^0|K_L^0\rangle e^{-i\omega_L t - \Gamma_L t/2}|^2 \\ &= \frac{1}{4}\left\{e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-\frac{\Gamma_S+\Gamma_L}{2}t}\cos\left[(\omega_S - \omega_L)t\right]\right\}. & (24)\end{aligned}$$

[1] Actually, this is only an assumption here. But it is a fundamental theorem in relativistic quantum mechanics that particle and anti-particle have the same mass (as well as the same total lifetime).

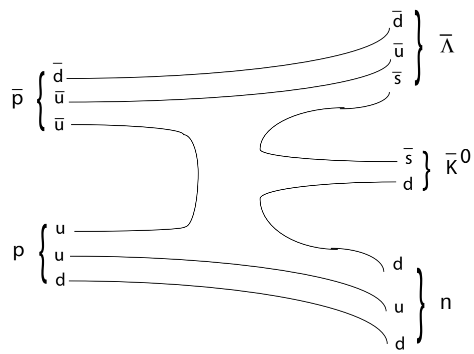


Figure 1: A possible reaction to produce an $\overline{K}^0$ meson. The lines indicate flow of quark flavors from left to right. No interactions are shown. Note that the production of the antibaryon tells us that it is a $\overline{K}^0$, not a $K^0$.

By measuring the frequency of the oscillation in the last term, we may measure the mass difference between the $K_S^0$ and the $K_L^0$. When the momentum is small, $\omega_S - \omega_L \approx m_S - m_L$. Because this difference is very small, it is experimentally intractable to attempt this with direct kinematic measurements. Measurements of the oscillation frequency yield a mass difference of

$$\begin{aligned} |m_S - m_L| &= 0.5 \times 10^{10}\ \mathrm{s}^{-1} \qquad (25) \\ &= \frac{0.5 \times 10^{10}\ \mathrm{s}^{-1}}{3 \times 10^{23}\ \mathrm{fm/s}} 200 \times 10^6\ \mathrm{eV\text{-}fm} \\ &= 3\ \mu\mathrm{eV}, \qquad (26) \end{aligned}$$

a difference comparable to the energy of a microwave photon. Since the mass of the kaon is approximately 500 MeV, this is a fractional difference of order one part in $10^{14}$!

We remark that this example shows that sometimes, even in non-relativistic quantum mechanics, the rest mass term in the energy must be included. This is because we may have a superpostion of states with different masses, and the time evolution of the components is correspondingly different, such that there is a time-dependent interference.

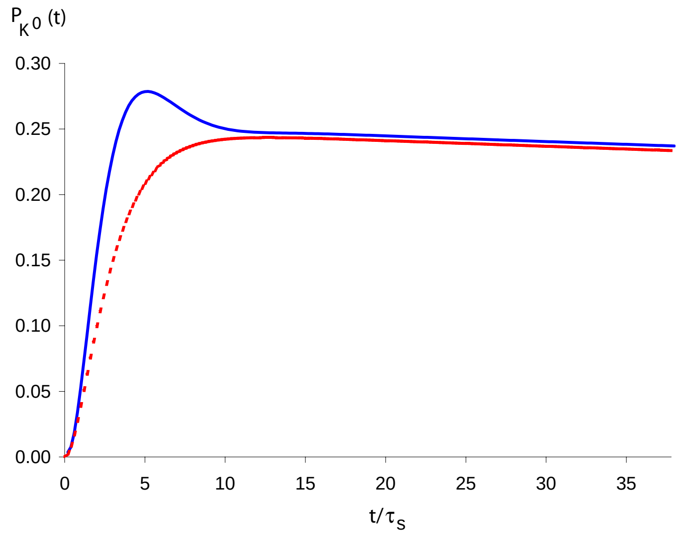


Figure 2: Upper curve: the $\overline{K}^0 \to K^0$ oscillation probability as a function of time (in units of $\tau_S$). Lower curve: the oscillation probability if $m_S = m_L$.

# 4 Exercises

1. Find the neutral kaon Hamiltonian in the $|K^0\rangle$, $|\overline{K}^0\rangle$ basis. Is the symmetry of your result consistent with the notion that the masses of particles and antiparticles are the same? Same question for the decay rates?

2. Repeat the derivation of Eqn. 24, but work in the density matrix formalism. We did not consider the possibility of decay when we develped this formalism, so be careful – you may find that you need to modify some of our discussion.

3. In this note, we discussed the neutral kaon $(K)$ meson, in particular the phenomenon of $K^0 - \bar{K}^0$ mixing. Let us think about this system a bit further. The $K^0$ and $\bar{K}^0$ mesons interact in matter, dominantly via the strong interaction. Approximately, the cross section for an interaction with a deuteron is:

$$\sigma(K^0 d) = 36\,\text{millibarns} \tag{27}$$
$$\sigma(\bar{K}^0 d) = 59\,\text{millibarns}, \tag{28}$$

at a kaon momentum of, say, 1.5 GeV. Note that a "barn" is a unit of area equal to $10^{-24}\,\text{cm}^2$.

(a) Consider a beam of kaons (momentum 1.5 GeV) incident on a target of liquid deuterium. Let $\lambda$ be the $K^0$ "interaction length", *i.e.*, the average distance that a $K^0$ will travel in the deuterium before it interacts according to the above cross section. Similarly, let $\bar{\lambda}$ be the $\bar{K}^0$ interaction length. To a good enough approximation for our purposes, you may treat the deuterium as a collection of deuterons (why?). The density of liquid deuterium is approximately $\rho = 0.17\,\text{g/cm}^3$. What are $\lambda$ and $\bar{\lambda}$, in centimeters?

(b) Suppose we have prepared a beam of $K_L^0$ mesons, *e.g.*, by first creating a $K^0$ beam and waiting long enough for the $K_S^0$ component to decay away. If we let this $K_L^0$ beam traverse a deuterium target, the $K^0$ and $\bar{K}^0$ components will interact differently, and we may end up with some $K_S^0$ mesons exiting the target. Let us make an estimate for the size of this effect.

Since the kaon is relativistic, we need to be a little careful compared with our discussion in the note: In the $K_L^0$ rest frame, the amplitude depends on time $t^*$ according to:

$$\exp(-im_L t^* - \Gamma_L t^*/2), \tag{29}$$

where $\Gamma_L = 1/\tau_L$ is the $K_L^0$ decay rate. In the laboratory frame, where the kaon is moving with speed $v$, and $\gamma = 1/\sqrt{1-v^2}$, $t^* \to t/\gamma$, where $t$ is the time as measured in the laboratory frame. In the lab frame, we have $t/\gamma = x/\gamma v$, and we may write the amplitude as for the $K_L^0$ as:

$$\exp(-im_L x/\gamma v - \Gamma_L x/2\gamma v), \tag{30}$$

Let us consider a deuterium target, of thickness $w$, along the beam direction. At a distance $x$ into the target, an interaction may occur, resulting in a final state:

$$\frac{1}{\sqrt{2}}(f|K^0\rangle - \bar{f}|\bar{K}^0\rangle), \tag{31}$$

where, for example, the amplitude $f$ for the $K^0$ component traversing distance $dx$ is just:

$$f = e^{-dx/2\lambda} \approx 1 - \frac{dx}{2\lambda}. \qquad (32)$$

Put all this together and find an expression for the probability to observe a $K_S^0$ to emerge from the deuterium, for a $K_L^0$ incident. Assume that $w \ll \lambda$. You may wish to use $\Delta m \equiv m_L - m_S$, $\Gamma_{S,L} \equiv 1/\tau_{S,L}$, and $\Delta\Gamma \equiv \Gamma_L - \Gamma_S \approx -\Gamma_S$

(c) Suppose $w = 10\,\text{cm}$ and $\gamma v = 3$. What is the probability to observe a $K_S^0$ emerging from the target? What is the probability to observe a $K_L^0$? You may use:

$$\Gamma_S = 1.1 \times 10^{10}\,\text{s}^{-1}, \qquad (33)$$

$$\Delta m = 0.5 \times 10^{10}\,\text{s}^{-1}. \qquad (34)$$

You have been investigating a phenomenon often called "regeneration" – by passing through material, a $K_S^0$ component to the beam has been "regenerated". A similar consideration has been proposed to help explain the "solar neutrino problem".