Physics 195a
Course Notes
The K^0: An Interesting Example of a "Two-State" System: Solutions to Exercises
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1 Exercises

1. Find the neutral kaon Hamiltonian in the $|K^0\rangle$, $|\overline{K}^0\rangle$ basis. Is the symmetry of your result consistent with the notion that the masses of particles and antiparticles are the same? Same question for the decay rates?

   Solution: In the $|K_S^0\rangle, |K_L^0\rangle$ basis, the Hamiltonian operator is:

$$H_{SL} = \begin{pmatrix} \omega_S - i\Gamma_S/2 & 0 \\ 0 & \omega_L - i\Gamma_L/2 \end{pmatrix}. \tag{1}$$

   We wish to make a basis change to the $|K^0\rangle$, $|\overline{K}^0\rangle$ basis:

$$H_{K\overline{K}} = R^{-1} H_{SL} R, \tag{2}$$

   where R is the matrix which transforms a vector in the $|K^0\rangle$, $|\overline{K}^0\rangle$ basis to one in the $|K_S^0\rangle, |K_L^0\rangle$ basis.

   The two bases are related by:

$$|K_S^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\overline{K}^0\rangle) \tag{3}$$

$$|K_L^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\overline{K}^0\rangle) \tag{4}$$

   The matrix which transforms a vector in the $|K^0\rangle, |\overline{K}^0\rangle$ basis to one in the $|K_S^0\rangle$, $|K_L^0\rangle$ is thus:

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{5}$$

   [Aside: this is a **Hadamard matrix** of order two. Roughly, a Hadamard matrix is an even matrix where each element can be 1 or -1, and pairs of rows or columns have half of their elements in common, and half opposite.]

Thus,

$$
\begin{aligned}
H_{K\overline{K}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \omega_S - i\Gamma_S/2 & 0 \\ 0 & \omega_L - i\Gamma_L/2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\
&= \frac{1}{2}\begin{pmatrix} \omega_S + \omega_L - i(\Gamma_S + \Gamma_L)/2 & \omega_S - \omega_L - i(\Gamma_S - \Gamma_L)/2 \\ \omega_S - \omega_L - i(\Gamma_S - \Gamma_L)/2 & \omega_S + \omega_L - i(\Gamma_S + \Gamma_L)/2 \end{pmatrix} \\
&= \begin{pmatrix} M - i\Gamma/2 & (\Delta M - i\Delta\Gamma/2)/2 \\ (\Delta M - i\Delta\Gamma/2)/2 & M - i\Gamma/2 \end{pmatrix}, \qquad (6)
\end{aligned}
$$

where $M \equiv (\omega_S + \omega_L)/2$, $\Gamma \equiv (\Gamma_s + \Gamma_L)/2$, $\Delta M \equiv M_S - M_L$, and $\Delta\Gamma \equiv \Gamma_S - \Gamma_L$.

2. Repeat the derivation of Eqn. 24, but work in the density matrix formalism. We did not consider the possibility of decay when we develped this formalism, so be careful – you may find that you need to modify some of our discussion.

   Solution: In the density matrix course note, we had the time evolution of the density matrix given by:

$$\frac{d}{dt}\rho(t) = -i[H, \rho]. \qquad (7)$$

   If we blindly use this equation in the present problem, we get a nonsensical result. The problem is that the K^0's are decaying, so the density matrix no longer corresponds to "conserved" probability content. Thus, we need to work out the time dependence anew. We repeat the steps in the density matrix note, but now without assuming $H = H^\dagger$:

$$
\begin{aligned}
\frac{d}{dt}\rho(t) &= \frac{d}{dt}\left[|\psi(t)\rangle\langle\psi(t)|\right] = \frac{d}{dt}\left\{|\psi(t)\rangle \left[|\psi(t)\rangle\right]^\dagger\right\} \\
&= \left[\frac{1}{i}H|\psi(t)\rangle\right]\langle\psi(t)| + |\psi(t)\rangle\left[\frac{1}{i}H|\psi(t)\rangle\right]^\dagger \\
&= \frac{1}{i}\left[H|\psi(t)\rangle\langle\psi(t)| - |\psi(t)\rangle\langle\psi(t)|H^\dagger\right] \\
&= -i\left[H\rho(t) - \rho(t)H^\dagger\right]. \qquad (8)
\end{aligned}
$$

   We are given that at time $t = 0$:

$$\psi(0) = |\overline{K}^0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad (9)$$

where the later representation is in the K_S, K_L basis. We'll work in this basis here, though that isn't required – we could use the result of exercise 1 and work in the $|K^0\rangle, |\overline{K}^0\rangle$ basis just as well.

We may write the time-dependent density matrix in the form:

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \begin{pmatrix} \alpha^*(t) & \beta^*(t) \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}. \quad (10)$$

Let's work out the time dependence, knowing that the Hamiltonian is

$$H_{SL} = \begin{pmatrix} \omega_S - i\Gamma_S/2 & 0 \\ 0 & \omega_L - i\Gamma_L/2 \end{pmatrix} \equiv \begin{pmatrix} S & 0 \\ 0 & L \end{pmatrix}. \quad (11)$$

We thus have:

$$\begin{aligned} \frac{d}{dt}\rho(t) &= \frac{1}{i}\left[\begin{pmatrix} S & 0 \\ 0 & L \end{pmatrix}\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} - \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}\begin{pmatrix} S^* & 0 \\ 0 & L^* \end{pmatrix}\right] \\ &= \frac{1}{i}\begin{pmatrix} (S-S^*)|\alpha|^2 & (S-L^*)\alpha\beta^* \\ (L-S^*)\alpha^*\beta & (L-L^*)|\beta|^2 \end{pmatrix} \\ &= \begin{pmatrix} -\Gamma_S|\alpha|^2 & -i(S-L^*)\alpha\beta^* \\ -i(L-S^*)\alpha^*\beta & -\Gamma_L|\beta|^2 \end{pmatrix}. \end{aligned} \quad (12)$$

We may integrate this equation to find:

$$\rho(t) = \begin{pmatrix} |\alpha(0)|^2 e^{-\Gamma_S t} & \alpha(0)\beta^*(0)e^{-i(S-L^*)t} \\ \alpha^*(0)\beta(0)e^{-i(L-S^*)t} & |\beta(0)|^2 e^{-\Gamma_L t} \end{pmatrix}. \quad (13)$$

Our initial condition is that $\alpha(0) = 1/\sqrt{2}$ and $\beta(0) = -1/\sqrt{2}$. Thus,

$$\rho(t) = \frac{1}{2}\begin{pmatrix} e^{-\Gamma_S t} & -e^{-i(S-L^*)t} \\ -e^{-i(L-S^*)t} & e^{-\Gamma_L t} \end{pmatrix}. \quad (14)$$

We wish to know the probability to find a K^0 at time t:

$$\begin{aligned} P_{K^0}(t) &= \mathrm{Tr}\left[\rho(t)|K^0\rangle\langle K^0|\right] \\ &= \mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} e^{-\Gamma_S t} & -e^{-i(S-L^*)t} \\ -e^{-i(L-S^*)t} & e^{-\Gamma_L t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] \\ &= \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - e^{-i(S-L^*)t} - e^{-i(L-S^*)t}\right] \\ &= \frac{1}{4}\left\{e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-\left(\frac{\Gamma_S+\Gamma_L}{2}\right)t}\cos\left[(\omega_S-\omega_L)t\right]\right\}. \end{aligned} \quad (15)$$

3. In this note, we discussed the neutral kaon (K) meson, in particular the phenomenon of $K^0 - \bar{K}^0$ mixing. Let us think about this system a bit further. The K^0 and $\bar{K}^0$ mesons interact in matter, dominantly via the strong interaction. Approximately, the cross section for an interaction with a deuteron is:

$$\sigma(K^0 d) = 36\,\text{millibarns} \tag{16}$$
$$\sigma(\bar{K}^0 d) = 59\,\text{millibarns}, \tag{17}$$

at a kaon momentum of, say, 1.5 GeV. Note that a "barn" is a unit of area equal to $10^{-24}\,\text{cm}^2$.

(a) Consider a beam of kaons (momentum 1.5 GeV) incident on a target of liquid deuterium. Let λ be the K^0 "interaction length", *i.e.*, the average distance that a K^0 will travel in the deuterium before it interacts according to the above cross section. Similarly, let $\bar{\lambda}$ be the $\bar{K}^0$ interaction length. To a good enough approximation for our purposes, you may treat the deuterium as a collection of deuterons (why?). The density of liquid deuterium is approximately $\rho = 0.17\,\text{g/cm}^3$. What are λ and $\bar{\lambda}$, in centimeters?

Solution: The mean distance to interact is given by the inverse of the effective size (cross section) presented by a scattering center, divided by the number density of scattering centers in the material:

$$\lambda = \frac{1}{\sigma \rho_{\#}}. \tag{18}$$

The mass of a deuterium atom is $m_d \sim 1876$ MeV, hence, the number density of scattering centers in liquid deuterium is

$$\rho_{\#} = \rho/m_d = \frac{0.17\,\text{g/cm}^3}{1876\,\text{MeV} \times 1.783 \times 10^{-27}\,\text{g/MeV}} = 5.08 \times 10^{22}\,\text{cm}^{-3}. \tag{19}$$

Thus,

$$\lambda = \frac{1}{36\,\text{mb} \times 10^{-27}\,\text{cm}^2/\text{mb} \times 5.08 \times 10^{22}\,\text{cm}^{-3}} = 550\,\text{cm} \tag{20}$$
$$\bar{\lambda} = 330\,\text{cm}. \tag{21}$$

(b) Suppose we have prepared a beam of K_L^0 mesons, *e.g.*, by first creating a K^0 beam and waiting long enough for the K_S^0 component to decay away. If we let this K_L^0 beam traverse a deuterium target, the K^0 and $\bar{K}^0$ components will interact differently, and we may end up with some K_S^0 mesons exiting the target. Let us make an estimate for the size of this effect.

Since the kaon is relativistic, we need to be a little careful compared with our discussion in the note: In the K_L^0 rest frame, the amplitude depends on time t^* according to:

$$\exp(-im_L t^* - \Gamma_L t^*/2), \tag{22}$$

where $\Gamma_L = 1/\tau_L$ is the K_L^0 decay rate. In the laboratory frame, where the kaon is moving with speed v, and $\gamma = 1/\sqrt{1-v^2}$, $t^* \rightarrow t/\gamma$, where t is the time as measured in the laboratory frame. In the lab frame, we have $t/\gamma = x/\gamma v$, and we may write the amplitude as for the K_L^0 as:

$$\exp(-im_L x/\gamma v - \Gamma_L x/2\gamma v), \tag{23}$$

Let us consider a deuterium target, of thickness w, along the beam direction. At a distance x into the target, an interaction may occur, resulting in a final state:

$$\frac{1}{\sqrt{2}}(f|K^0\rangle - \bar{f}|\bar{K}^0\rangle), \tag{24}$$

where, for example, the amplitude f for the K^0 component traversing distance dx is just:

$$f = e^{-dx/2\lambda} \approx 1 - \frac{dx}{2\lambda}. \tag{25}$$

Put all this together and find an expression for the probability to observe a K_S^0 to emerge from the deuterium, for a K_L^0 incident. Assume that $w \ll \lambda$. You may wish to use $\Delta m \equiv m_L - m_S$, $\Gamma_{S,L} \equiv 1/\tau_{S,L}$, and $\Delta\Gamma \equiv \Gamma_L - \Gamma_S \approx -\Gamma_S$

Solution: We'll use distance x as surrogate for time t, with $x = t = 0$ at the entrance to the deuterium target. We'll work in the

$|K_S^0\rangle, |K_L^0\rangle$ basis here. There are two pieces to the Hamiltionian to worry about now.

First, there is the weak interaction piece we have already been discussing:

$$H_W = \begin{pmatrix} S & 0 \\ 0 & L \end{pmatrix}, \tag{26}$$

where

$$\begin{aligned} S &\equiv (m_S - i\Gamma_S/2)/(\gamma v) \quad (27) \\ L &\equiv (m_L - i\Gamma_L/2)/(\gamma v). \quad (28) \end{aligned}$$

Note that we have defined things so that the Hamiltonian is now the operator id_x.

The other piece of the Hamiltonian, H_d describes the strong interaction with the deuterium. It "takes away" bits of our wave function, at different rates for the K^0 and $\overline{K}^0$ components. Suppose, for example, we start with a pure K_L^0 state, and let it traverse a small distance dx in the deuterium. The state becomes modified according to:

$$\begin{aligned} |\psi(dx)\rangle &= \frac{1}{\sqrt{2}}(f|K^0\rangle - \bar{f}|\bar{K}^0\rangle) \\ &= \frac{1}{\sqrt{2}}\left[\left(1 - \frac{dx}{2\lambda}\right)|K^0\rangle - \left(1 - \frac{dx}{2\bar{\lambda}}\right)|\overline{K}^0\rangle\right] \quad (29) \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d\psi}{dx} &= -\frac{1}{2\sqrt{2}}\left[\frac{1}{\lambda}|K^0\rangle - \frac{1}{\bar{\lambda}}|\overline{K}^0\rangle\right] \\ &= -\frac{1}{4}\left[\left(\frac{1}{\lambda} + \frac{1}{\bar{\lambda}}\right)|K_L^0\rangle + \left(\frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\right)|K_S^0\rangle\right] \\ &= -a|K_L^0\rangle + b|K_S^0\rangle, \quad (30) \end{aligned}$$

where

$$\begin{aligned} a &\equiv \frac{1}{4}\left(\frac{1}{\lambda} + \frac{1}{\bar{\lambda}}\right) \quad (31) \\ b &\equiv \frac{1}{4}\left(\frac{1}{\bar{\lambda}} - \frac{1}{\lambda}\right) \quad (32) \end{aligned}$$

For an initial K_S^0 we similarly obtain:

$$\frac{d\psi}{dx} = -a|K_S^0\rangle + b|K_L^0\rangle. \tag{33}$$

Hence, the Hamiltonian driving this transformation is:

$$H_d = i\frac{d}{dx} = \begin{pmatrix} -ia & ib \\ ib & -ia \end{pmatrix}. \tag{34}$$

Note that the diagonal elements are pure imaginary, similar to the decay terms in H_W. This reflects the fact that kaons are actually being removed in the scattering. Let's re-express the Hamiltonian in the form $H = H_0 + H_1$, where

$$H_0 = \begin{pmatrix} S' & 0 \\ 0 & L' \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 0 & ib \\ ib & 0 \end{pmatrix}, \tag{35}$$

where

$$\begin{aligned} S' &= (m_S - i\Gamma_S'/2)/\gamma v, & (36)\\ L' &= (m_L - i\Gamma_L'/2)/\gamma v, & (37)\\ \Gamma_S' &= \Gamma_S + 2\gamma v a, & (38)\\ \Gamma_L' &= \Gamma_L + 2\gamma v a. & (39) \end{aligned}$$

Thus, we have absorbed the a terms into effective decay rates for the kaons.

The Schrödinger equation we wish to solve is:

$$i\frac{d}{dx}\psi(x) = (H_0 + H_1)\psi(x). \tag{40}$$

Explicitly, if the K_S^0 component is α and the K_L^0 component is β:

$$i\frac{d}{dx}\begin{pmatrix} \alpha(x) \\ \beta(x) \end{pmatrix} = \begin{pmatrix} S'\alpha + ib\beta \\ L'\beta + ib\alpha \end{pmatrix}. \tag{41}$$

Let's see if we can solve this pair of coupled equations.

The form of these equations suggests we consider a solution of the form:

$$\begin{aligned} \alpha(x) &= f(x)e^{-iS'x} & (42)\\ \beta(x) &= g(x)e^{-iL'x}. & (43) \end{aligned}$$

We substitute these back into the differential equations, and find that we may eliminate $g(x)$ and obtain a homogeneous equation for $f(x)$:

$$f'' - i(S' - L')f' - b^2 f = 0. \tag{44}$$

The solution which satisfies the boundary condition $f(0) = 0$ is:

$$f(x) = Ae^{-i\frac{L'-S'}{2}x}\left\{\exp\left[\frac{i}{2}\sqrt{(S'-L')^2 - 4b^2}x\right] - \exp\left[-\frac{i}{2}\sqrt{(S'-L')^2 - 4b^2}x\right]\right\}. \tag{45}$$

Thus,

$$\alpha(x) = Ae^{-i\frac{L'+S'}{2}x}\left\{\exp\left[\frac{i}{2}\sqrt{(S'-L')^2 - 4b^2}x\right] - \exp\left[-\frac{i}{2}\sqrt{(S'-L')^2 - 4b^2}x\right]\right\}. \tag{46}$$

We can substitute this back into the equation for β:

$$\begin{aligned} \beta(x) &= \frac{1}{b}\left(\frac{d\alpha}{dx} + iS'\alpha\right) \\ &= \frac{iA}{2b}e^{-i\frac{L'+S'}{2}x}\left\{(L'-S')\left[e^{\frac{i}{2}R} - e^{-\frac{i}{2}R}\right] + R\left[e^{\frac{i}{2}R} + e^{-\frac{i}{2}R}\right]\right\}, \end{aligned} \tag{47}$$

where

$$R \equiv \sqrt{(S'-L')^2 - 4b^2}. \tag{48}$$

The boundary condition is that

$$1 = \beta(0) = \frac{iA}{b}R, \tag{49}$$

hence

$$A = -ib/R. \tag{50}$$

The probability of seeing a K_S^0 emerging from the deuterium is thus

$$\begin{aligned} P_{K_S^0}(w) &= |\alpha(w)|^2 \\ &= \frac{b^2}{|R|^2}e^{w\Im(L'+S')}\left[e^{\frac{i}{2}Rw}e^{-\frac{i}{2}R^*w} + e^{-\frac{i}{2}Rw}e^{\frac{i}{2}R^*w} - e^{\frac{i}{2}Rw}e^{\frac{i}{2}R^*w} - e^{-\frac{i}{2}Rw}e^{-\frac{i}{2}R^*w}\right] \\ &= \frac{b^2}{|R|^2}e^{w\Im(L'+S')}\left[e^{-\frac{w}{2}\Im R} + e^{+\frac{w}{2}\Im R} - 2\cos(w\Re R)\right]. \end{aligned} \tag{51}$$

Let's rewrite R in terms of the physical inputs:

$$\begin{aligned} R &= \sqrt{(S'-L')^2-4b^2} \\ &= \sqrt{\left[(m_S-i\Gamma'_S/2)-(m_L-i\Gamma'_L/2)\right]/(\gamma v)^2-4b^2} \\ &= \sqrt{(\Delta m-i\Delta\Gamma/2)^2/(\gamma v)^2-4b^2}. \end{aligned} \tag{52}$$

Hence,

$$\begin{aligned} |R|^2 &= |(\Delta m-i\Delta\Gamma/2)^2/(\gamma v)^2-4b^2| \\ &= |(\Delta m)^2-(\Delta\Gamma/2)^2-(2b\gamma v)^2-i\Delta m\Delta\Gamma|/(\gamma v)^2 \\ &= \sqrt{\left[(\Delta m)^2+(\Delta\Gamma/2)^2\right]^2-2(2\gamma vb)^2\left[(\Delta m)^2-(\Delta\Gamma/2)^2\right]+2\gamma vb)^4}/(\gamma v)^2 \\ &\approx \left[(\Delta m)^2+(\Delta\Gamma/2)^2\right]/(\gamma v)^2, \end{aligned} \tag{53}$$

where the approximation is for $(\gamma vb)^2$ small. Note that we have not formerly made any such approximations, so our result may be applied in situations where this is not valid. However, it is valid here, and we now proceed to use the approximation that b is small. We note that this corresponds to assuming that there is at most one interaction in the target.

In this approximation, $R \approx (\Delta m - i\Delta\Gamma/2)/(\gamma v)$. We finally have:

$$P_{K_S^0}(w) \approx \frac{b^2(\gamma v)^2}{(\Delta m)^2+(\Delta\Gamma/2)^2}e^{-2aw}\left[e^{-\frac{\Gamma_L}{\gamma v}w}+e^{-\frac{\Gamma_S}{\gamma v}w}-2e^{-\frac{\Gamma_L+\Gamma_S}{2\gamma v}w}\cos\left(\frac{\Delta m w}{\gamma v}\right)\right]. \tag{54}$$

Likewise, the probability to observe a K_L^0 emerging from the target is:

$$\begin{aligned} P_{K_L^0}(w) &= |\beta(w)|^2 \\ &= e^{-2aw}e^{-\frac{\Gamma_L}{\gamma v}w}. \end{aligned} \tag{55}$$

(c) Suppose $w = 10\,\text{cm}$ and $\gamma v = 3$. What is the probability to observe a K_S^0 emerging from the target? What is the probability to observe a K_L^0? You may use:

$$\Gamma_S = 1.1\times 10^{10}\,\text{s}^{-1}, \tag{56}$$

$$\Delta m = 0.5\times 10^{10}\,\text{s}^{-1}. \tag{57}$$

Solution: I get:

$$P_{K_S^0}(w) \approx 4.9 \times 10^{-6} \tag{58}$$
$$P_{K_L^0}(w) \approx 0.974. \tag{59}$$

You have been investigating a phenomenon often called "regeneration" – by passing through material, a K_S^0 component to the beam has been "regenerated". A similar consideration has been proposed to help explain the "solar neutrino problem".