Physics 195a
Course Notes
The Simple Harmonic Oscillator: Creation and Destruction Operators: Solutions to Exercises
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1 Exercises

1. We have noticed some things about the qualitative behavior of wave functions in our discussion of Fig. **??**. Consider the one-dimensional problem with potential function given by:

$$V(x) = \begin{cases} 0 & \text{for } |x| \leq a, \\ V_0 & \text{for } |x| > a, \end{cases} \tag{1}$$

where $V_0 > 0$ and $a > 0$.

(a) Suppose that there are four bound states. Make a qualitative, but careful, drawing of what you expect the first four wave functions to look like, in the spirit of Fig. **??**.

Solution:

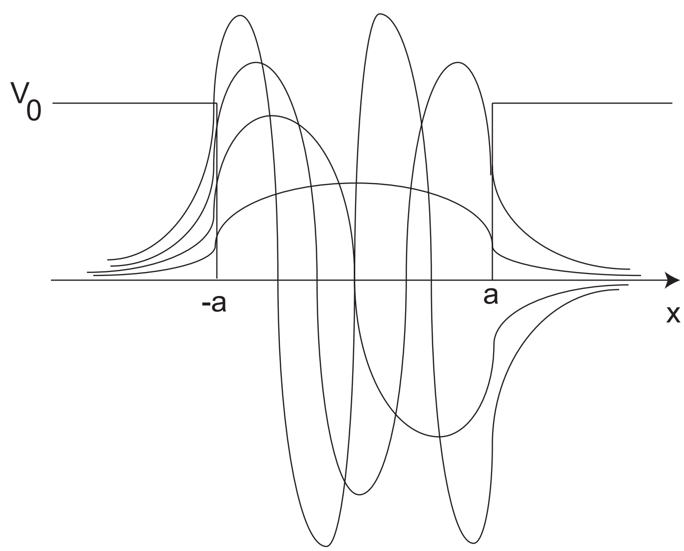

Figure 1: Qualitative wave functions for bound states.

(b) Make a qualitative drawing for the wave function of a state with energy above V_0.

Solution:

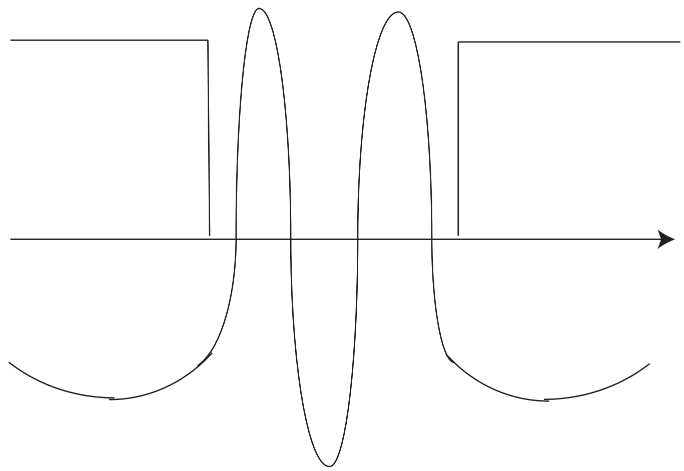

Figure 2: Qualitative wave function for continuum state.

2. Let us generalize the discussion of the simple harmonic oscillator to three dimensions. In this case, the Hamiltonian is:

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}m(\Omega\mathbf{x})^2, \tag{2}$$

where Ω is a 3×3 symmetric real matrix.

(a) Determine the energy spectrum and eigenvectors of this system.

Solution: The matrix Ω is diagonalizable by an orthogonal matrix (a rotation), so let:

$$R\Omega R^T = \begin{pmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix}, \tag{3}$$

and let $\mathbf{y} = R\mathbf{x}$. The kinetic energy piece is a scalar, independent of what coordinate basis we use. Hence, we may rewrite our Hamiltonian as:

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}m(\omega_1^2 y_1^2 + \omega_2^2 y_2^2 + \omega_3^2 y_3^2). \tag{4}$$

(b) Suppose the potential is spherically symmetric. Using the equivalent one-dimensional potential approach, find the eigenvalues and eigenvectors of H corresponding to the possible values of orbital angular momentum.

Solution: The spherically symmetric Hamiltonian may be written in the form:

$$H = \frac{1}{2m}\mathbf{p}^2 + \frac{1}{2}m\omega^2\mathbf{x}^2, \tag{5}$$

Letting $\psi_{n\ell}(\mathbf{x}) = \frac{u_{n\ell}(r)}{r}Y_{\ell m}(\theta, \phi)$, we have the equivalent one-dimensional Schrödinger equation:

$$\left[-\frac{1}{2m}\frac{d^2}{dr^2} + \frac{1}{2}m\omega^2 r^2 + \frac{\ell(\ell+1)}{2mr^2}\right]u_{n\ell}(r) = Eu_{n\ell}(r). \tag{6}$$

We put the equation in a more convenient dimensionless form. Let

$$k \equiv \sqrt{2mE}, \tag{7}$$

$$\rho \equiv kr, \tag{8}$$

$$\lambda \equiv \left(\frac{m\omega}{k^2}\right)^2. \tag{9}$$

Then the Schrödinger equation may be written:

$$\left[\frac{d^2}{d\rho^2} - \lambda\rho^2 - \frac{\ell(\ell+1)}{\rho^2} + 1\right]v(\rho) = 0, \tag{10}$$

where $v(\rho) = u_{n\ell}(\rho/k)$.

The asymptotic form of the Schrödinger equation is

$$\left(\frac{d^2}{d\rho^2} - \lambda\rho^2\right)v(\rho) = 0, \tag{11}$$

which suggests we try a solution with the asymptotic form:

$$v(\rho) = f(\rho)e^{-\sqrt{\lambda}\rho^2/2}. \tag{12}$$

Near $\rho = 0$, the equation is of the form:

$$\left(\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2}\right)v(\rho) = 0. \tag{13}$$

Thus, we try a solution of the form:

$$v(\rho) = \rho^{\ell+1} g(\rho) e^{-\sqrt{\lambda}\rho^2/2}, \tag{14}$$

where we will look for a series solution for $g(\rho)$.

Substituting the above form into the Schrödinger equation, we obtain the following differenial equation for $g(\rho)$:

$$\rho g'' + 2(\ell + 1 - \sqrt{\lambda}\rho^2) g' + \left[1 - \sqrt{\lambda}(2\ell + 3)\right] \rho g = 0. \tag{15}$$

Letting

$$g(\rho) = \sum_{j=0} c_j \rho^j, \tag{16}$$

we find that $c_j = 0$ for j odd, and for even j we have the recurrence relation:

$$c_{j+2} = c_j \frac{1 - \sqrt{\lambda}(2\ell + 2j + 3)}{(j+2)(2\ell + j + 3)}. \tag{17}$$

We already have the asymptotic behavior, so we suppose that the series stops at $j = n$, *i.e.*,

$$1 = \sqrt{\lambda}(2\ell + 2n + 3). \tag{18}$$

Thus, we have the discrete energy spectrum:

$$E_{n\ell} = \frac{\omega}{2}(2\ell + 2n + 3) = \omega\left(n + \ell + \frac{3}{2}\right). \tag{19}$$

Note that n is even.

We may now substitute $\sqrt{\lambda} = 1/(2\ell + 2n + 3)$ and use the recurrence relation to obtain an expression for coefficient c_j:

$$\begin{aligned} c_j &= c_{j-2} \frac{2(n - j + 2)}{(2n + 2\ell + 3) j (2\ell + j + 1)} & (20) \\ &= c_0 \frac{2}{(2n + 2\ell + 3)} \frac{(n - j + 2)(n - j + 4)(n - j + 6) \cdots (n - j + j)}{j(j-2) \cdots 2 (2\ell + j + 1)(2\ell + j - 1) \cdots (2\ell + 3)} \\ &= c_0 \frac{2}{(2n + 2\ell + 3)} \frac{n!!}{(n - j)!!} \frac{1}{j!!} \frac{(2\ell + 1)!!}{(2\ell + j + 1)!!} & (21) \end{aligned}$$

Putting it all together, we have:

$$\psi_{n\ell m}(\mathbf{x}) = \frac{u_{n\ell}(r)}{r} Y_{\ell m}(\theta, \phi) \tag{22}$$

$$= \frac{v_{n\ell}(kr)}{r} Y_{\ell m}(\theta, \phi) \tag{23}$$

$$= \frac{(kr)^{\ell+1}}{r} \exp\left[-\frac{1}{2}\frac{(kr)^2}{2(n+\ell+3/2)}\right] Y_{\ell m}(\theta, \phi)$$
$$c_{0n\ell} \sum_{j=0}^{n} \frac{2}{(2n+2\ell+3)} \frac{n!!}{(n-j)!!} \frac{1}{j!!} \frac{(2\ell+1)!!}{(2\ell+j+1)!!} (kr)^j \tag{24}$$

where $c_{0n\ell}$ is determined by normalization.