Physics 125
Course Notes
Angular Momentum
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1 Introduction

This note constitutes a discussion of angular momentum in quantum mechanics. Several results are obtained, important to understanding how to solve problems involving rotational symmetries in quantum mechanics. We will often give only partial proofs to the theorems, with the intent that the reader complete them as necessary. In the hopes that this will prove to be a useful reference, the discussion is rather more extensive than usually encountered in quantum mechanics textbooks.

2 Rotations: Conventions and Parameterizations

A rotation by angle θ about an axis $\mathbf{e}$ (passing through the origin in R^3) is denoted by $R_{\mathbf{e}}(\theta)$. We'll denote an abstract rotation simply as R. It is considered to be "positive" if it is performed in a clockwise sense as we look along $\mathbf{e}$. As with other transformations, our convention is that we think of a rotation as a transformation of the state of a physical system, and not as a change of coordinate system (sometimes referred to as the "active view"). If $R_{\mathbf{e}}(\theta)$ acts on a configuration of points ("system") we obtain a new, rotated, configuration: If $\mathbf{x}$ is a point of the old configuration, and $\mathbf{x}'$ is its image under the rotation, then:

$$\mathbf{x}' = R_{\mathbf{e}}(\theta)\mathbf{x}. \tag{1}$$

That is, R is a linear transformation described by a 3×3 real matrix, relative to a basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

Geometrically, we may observe that:

$$\begin{aligned}
R_{\mathbf{e}}(-\theta) &= R_{\mathbf{e}}^{-1}(\theta), && (2)\\
I &= R_{\mathbf{e}}(0) = R_{\mathbf{e}}(2\pi n), \quad n = \text{integer}, && (3)\\
R_{\mathbf{e}}(\theta)R_{\mathbf{e}}(\theta') &= R_{\mathbf{e}}(\theta+\theta'), && (4)\\
R_{\mathbf{e}}(\theta+2\pi n) &= R_{\mathbf{e}}(\theta), \quad n = \text{integer}, && (5)\\
R_{-\mathbf{e}} &= R_{\mathbf{e}}(-\theta). && (6)
\end{aligned}$$

A product of the form $R_{\mathbf{e}}(\theta)R_{\mathbf{e}'}(\theta')$ means "first do $R_{\mathbf{e}'}(\theta')$, then do $R_{\mathbf{e}}(\theta)$ to the result". All of these identities may be understood in terms of matrix identities, in addition to geometrically. Note further that the set of all rotations about a fixed axis $\mathbf{e}$ forms a one-parameter abelian group.

It is useful to include in our discussion of rotations the notion also of reflections: We'll denote the space reflection with respect to the origin by P, for **parity**:

$$P\mathbf{x} = -\mathbf{x}. \tag{7}$$

Reflection in a plane through the origin is called a mirroring. Let $\mathbf{e}$ be a unit normal to the mirror plane. Then

$$M_{\mathbf{e}}\mathbf{x} = \mathbf{x} - 2\mathbf{e}(\mathbf{e}\cdot\mathbf{x}), \tag{8}$$

since the component of the vector in the plane remains the same, and the normal component is reversed.

Theorem: 1.

$$[P, R_{\mathbf{e}}(\theta)] = 0. \tag{9}$$
$$[P, M_{\mathbf{e}}] = 0. \tag{10}$$

(The proof of this is trivial, since $P = -I$.)

2.

$$PM_{\mathbf{e}} = M_{\mathbf{e}}P = R_{\mathbf{e}}(\pi). \tag{11}$$

3. P, $M_{\mathbf{e}}$, and $R_{\mathbf{e}}(\pi)$ are "involutions", that is:

$$P^2 = I. \tag{12}$$
$$M_{\mathbf{e}}^2 = I. \tag{13}$$
$$[R_{\mathbf{e}}(\pi)]^2 = I. \tag{14}$$

Theorem: Let $R_{\mathbf{e}}(\theta)$ be a rotation and let $\mathbf{e}'$, $\mathbf{e}''$ be two unit vectors perpendicular to unit vector $\mathbf{e}$ such that $\mathbf{e}''$ is obtained from $\mathbf{e}'$ according to:

$$\mathbf{e}'' = R_{\mathbf{e}}(\theta/2)\mathbf{e}'. \tag{15}$$

Then

$$R_{\mathbf{e}}(\theta) = M_{\mathbf{e}''}M_{\mathbf{e}'} = R_{\mathbf{e}''}(\pi)R_{\mathbf{e}'}(\pi). \tag{16}$$

Hence, every rotation is a product of two mirrorings, and also a product of two rotations by π.

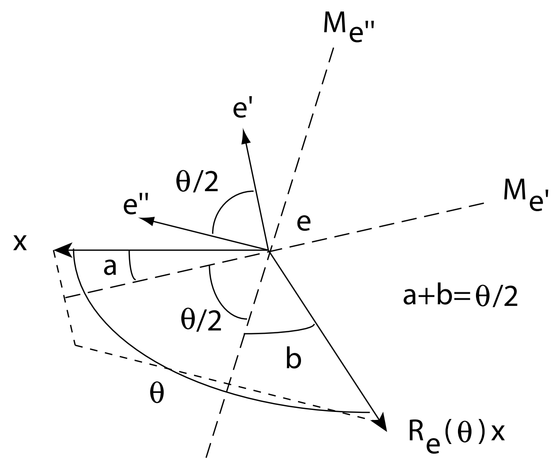

Figure 1: Proof of the theorem that a rotation about $\mathbf{e}$ by angle θ is equivalent to the product of two mirrorings or rotations by π.

Proof: We make a graphical proof, referring to Fig. 1.

Theorem: Consider the spherical triangle in Fig. 2.

1. We have
$$R_{\mathbf{e}_3}(2\alpha_3)R_{\mathbf{e}_2}(2\alpha_2)R_{\mathbf{e}_1}(2\alpha_1) = I, \tag{17}$$
where the unit vectors $\{\mathbf{e_i}\}$ are as labelled in the figure.

2. Hence, the product of two rotations is a rotation:
$$R_{\mathbf{e}_2}(2\alpha_2)R_{\mathbf{e}_1}(2\alpha_1) = R_{\mathbf{e}_3}(-2\alpha_3). \tag{18}$$
The set of all rotations is a group, where group multiplication is application of successive rotations.

Proof: Use the figure and label M_1 the mirror plane spanned by $\mathbf{e}_2, \mathbf{e}_3$, *etc.* Then we have:

$$\begin{aligned} R_{\mathbf{e}_1}(2\alpha_1) &= M_3 M_2 \\ R_{\mathbf{e}_2}(2\alpha_2) &= M_1 M_3 \\ R_{\mathbf{e}_3}(2\alpha_3) &= M_2 M_1. \end{aligned} \tag{19}$$

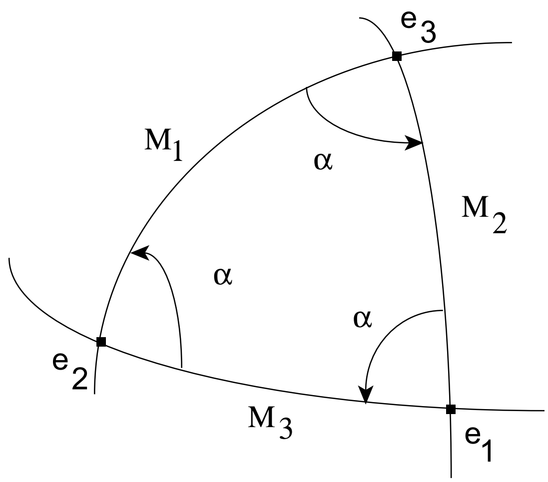

Figure 2: Illustration for theorem. Spherical triangle vertices are defined as the intersections of unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ on the surface of the unit sphere.

Thus,

$$R_{\mathbf{e}_3}(2\alpha_3)R_{\mathbf{e}_2}(2\alpha_2)R_{\mathbf{e}_1}(2\alpha_1) = (M_2M_1)(M_1M_3)(M_3M_2) = I. \tag{20}$$

The combination of two rotations may thus be expressed as a problem in spherical trigonometry.

As a corollary to this theorem, we have the following generalization of the addition formula for tangents:

Theorem: If $R_{\mathbf{e}}(\theta) = R_{\mathbf{e}''}(\theta'')R_{\mathbf{e}'}(\theta')$, and defining:

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{e}\tan\theta/2 \\ \boldsymbol{\tau}' &= \mathbf{e}'\tan\theta'/2 \\ \boldsymbol{\tau}'' &= \mathbf{e}''\tan\theta''/2, \end{aligned} \tag{21}$$

then

$$\boldsymbol{\tau} = \frac{\boldsymbol{\tau}' + \boldsymbol{\tau}'' + \boldsymbol{\tau}'' \times \boldsymbol{\tau}'}{1 - \boldsymbol{\tau}' \cdot \boldsymbol{\tau}''}. \tag{22}$$

This will be left as an exercise for the reader to prove.

Theorem: The most general mapping $\mathbf{x} \to \mathbf{x}'$ of R^3 into itself, such that the origin is mapped into the origin, and such that all distances are preserved, is a linear, real orthogonal transformation Q:

$$\mathbf{x}' = Q\mathbf{x}, \quad \text{where} \quad Q^T Q = I, \quad \text{and} \quad Q^* = Q. \tag{23}$$

Hence,

$$\mathbf{x}' \cdot \mathbf{y}' = \mathbf{x} \cdot \mathbf{y} \quad \forall\, \text{points}\, \mathbf{x}, \mathbf{y} \in R^3. \tag{24}$$

For such a mapping, either:

1. $\det(Q) = 1$, Q is called a **proper** orthogonal transformation, and is in fact a rotation. In this case,

$$\mathbf{x}' \times \mathbf{y}' = (\mathbf{x} \times \mathbf{y})' \quad \forall\, \text{points}\, \mathbf{x}, \mathbf{y} \in R^3. \tag{25}$$

or,

2. $\det(Q) = -1$, Q is called an **improper** orthogonal transformation, and is the product of a reflection (parity) and a rotation. In this case,

$$\mathbf{x}' \times \mathbf{y}' = -(\mathbf{x} \times \mathbf{y})' \quad \forall\, \text{points}\, \mathbf{x}, \mathbf{y} \in R^3. \tag{26}$$

The set of all orthogonal transformations on three dimensions forms a group (denoted $O(3)$), and the set of all proper orthogonal transformations forms a subgroup ($O^+(3)$ or $SO(3)$ of $O(3)$), in $1:1$ correspondence with, hence a "representation" of, the set of all rotations.

Proof of this theorem will be left to the reader.

3 Some Useful Representations of Rotations

Theorem: We have the following representations of rotations ($\mathbf{u}$ is a unit vector):

$$R_{\mathbf{u}}(\theta)\mathbf{x} = \mathbf{u}\mathbf{u} \cdot \mathbf{x} + [\mathbf{x} - \mathbf{u}\mathbf{u} \cdot \mathbf{x}] \cos\theta + \mathbf{u} \times \mathbf{x} \sin\theta, \tag{27}$$

and

$$R_{\mathbf{u}}(\theta) = e^{\theta \mathbf{u} \cdot \boldsymbol{\mathcal{J}}} = I + (\mathbf{u} \cdot \boldsymbol{\mathcal{J}})^2 (1 - \cos\theta) + \mathbf{u} \cdot \boldsymbol{\mathcal{J}} \sin\theta, \tag{28}$$

where $\boldsymbol{\mathcal{J}} = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ with:

$$\mathcal{J}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

Proof: The first relation may be seen by geometric inspection: It is a decomposition of the rotated vector into components along the axis of rotation, and the two orthogonal directions perpendicular to the axis of rotation.

The second relation may be demonstrated by noticing that $\mathcal{J}_i\mathbf{x} = \mathbf{e}_i \times \mathbf{x}$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the three basis unit vectors. Thus,

$$(\mathbf{u} \cdot \boldsymbol{\mathcal{J}})\mathbf{x} = \mathbf{u} \times \mathbf{x}, \tag{30}$$

and

$$(\mathbf{u} \cdot \boldsymbol{\mathcal{J}})^2\mathbf{x} = \mathbf{u} \times (\mathbf{u} \times \mathbf{x}) = \mathbf{u}(\mathbf{u} \cdot \mathbf{x}) - \mathbf{x}. \tag{31}$$

Further,

$$(\mathbf{u} \cdot \boldsymbol{\mathcal{J}})^{2n+m} = (-)^n(\mathbf{u} \cdot \boldsymbol{\mathcal{J}})^m, \quad n = 1, 2, \ldots; \; m = 1, 2. \tag{32}$$

The second relation then follows from the first.

Note that

$$\mathrm{Tr}\left[R_{\mathbf{u}}(\theta)\right] = \mathrm{Tr}\left[I + (\mathbf{u} \cdot \boldsymbol{\mathcal{J}})^2(1 - \cos\theta)\right]. \tag{33}$$

With

$$\mathcal{J}_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \mathcal{J}_2^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \mathcal{J}_3^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{34}$$

we have

$$\mathrm{Tr}\left[R_{\mathbf{u}}(\theta)\right] = 3 - 2(1 - \cos\theta) = 1 + 2\cos\theta. \tag{35}$$

This is in agreement with the eigenvalues of $R_{\mathbf{u}}(\theta)$ being $1, e^{i\theta}, e^{-i\theta}$.

Theorem: (**Euler parameterization**) Let $R \in O^+(3)$. Then R can be represented in the form:

$$R = R(\psi, \theta, \phi) = R_{\mathbf{e}_3}(\psi) R_{\mathbf{e}_2}(\theta) R_{\mathbf{e}_3}(\phi), \tag{36}$$

where the **Euler angles** ψ, θ, ϕ can be restricted to the ranges:

$$0 \leq \psi < 2\pi; \quad 0 \leq \theta \leq \pi; \quad 0 \leq \phi < 2\pi. \tag{37}$$

With these restrictions, the parameterization is unique, unless R is a rotation about $\mathbf{e}_3$, in which case $R_{\mathbf{e}_3}(\alpha) = R(\alpha - \beta, 0, \beta)$ for any β.

Proof: We refer to Fig. 3 to guide us. Let $\mathbf{e}'_k = R\mathbf{e}_k,\ k = 1, 2, 3$, noting that it is sufficient to consider the transformation of three orthogonal unit vectors, which we might as well take to be intitially along the basis directions. We must show that we can orient $\mathbf{e}'_k$ in any desired direction in order to prove that a general rotation can be described as asserted in the theorem.

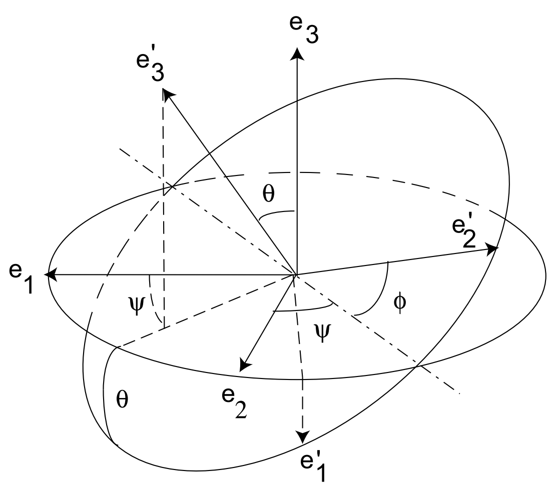

Figure 3: Illustration for visualizing the Euler angle theorem. It may be useful to think of the figure as the action of R on a “rigid body”, a unit disk with a unit normal attached, and two orthogonal unit vectors attached in the plane of the disk. The original and final positions of the disk and attached unit vectors are shown.

We note that $\mathbf{e}_3'$ does not depend on ϕ since this first rotation is about $\mathbf{e}_3$ itself. The polar angles of $\mathbf{e}_3'$ are given precisely by θ and ψ. Hence θ and ψ are uniquely determined (within the specified ranges, and up to the ambiguous case mentioned in the theorem) by $\mathbf{e}_3' = R\mathbf{e}_3$, which can be specified to any desired orientation. The angle ϕ is then determined uniquely by the orientation of the pair $(\mathbf{e}_1', \mathbf{e}_2')$ in the plane perpendicular to $\mathbf{e}_3'$.

We note that the rotation group $[O^+(3)]$ is a group of infinite order (or, is an "infinite group", for short). There are also an infinite number of subgroups of $O^+(3)$, including both finite and infinite subgroups. Some of the important finite subgroups may be classified as:

1. The **Dihedral** groups, D_n, corresponding to the proper symmetries of an n-gonal prism. For example, $D_6 \subset O^+(3)$ is the group of rotations which leaves a hexagonal prism invariant. This is a group of order 12, generated by rotations $R_{\mathbf{e}_3}(2\pi/6)$ and $R_{\mathbf{e}_2}(\pi)$.

2. The symmetry groups of the regular solids:

   (a) The symmetry group of the tetrahedron.

   (b) The symmetry group of the octahedron, or its "dual" (replace vertices by faces, faces by vertices) the cube.

   (c) The symmetry group of the icosahedron, or its dual, the dodecahedron.

   We note that the tetrahedron is self-dual.

An example of an infinite subgroup of $O^+(3)$ is D_∞, the set of all rotations which leaves a circular disk invariant, that is, including all rotations about the normal to the disk, and rotations by π about any axis in the plane of the disk.

4 Special Unitary Groups

The set of all $n \times n$ unitary matrices forms a group (under normal matrix multiplication), denoted by $U(n)$. $U(n)$ includes as a subgroup, the set of all $n \times n$ unitary matrices with determinant equal to 1 ("unimodular", or "special". This subgroup is denoted by $SU(n)$, for "Special Unitary" group.

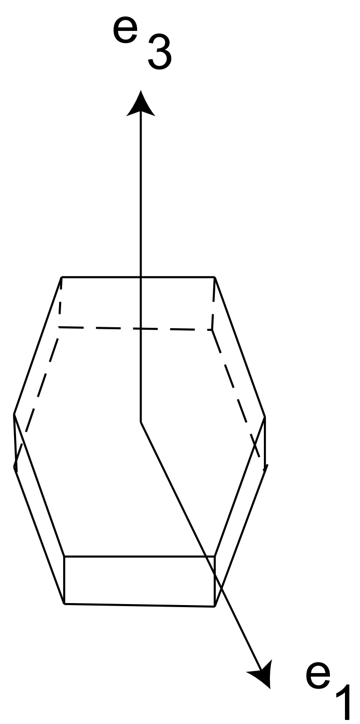

Figure 4: A hexagonal prism, to illustrate the group D_6.

The group of 2×2 unimodular unitary matrices, $SU(2)$, has a special connection with $O^+(3)$, which is very important in quantum mechanics. Consider the real vector space of all 2×2 traceless hermitian matrices, which we denote by V_3. The "3" refers to the fact that this is a three-dimensional vector space (even though it consists of 2×2 matrices). Hence, it can be put into $1:1$ correspondence with Euclidean 3-space, R^3. We may make this correspondence an isometry by introducing a positive-definite symmetric scalar product on V_3:

$$(X, Y) = \frac{1}{2}\mathrm{Tr}(XY), \quad \forall X, Y \in V_3. \tag{38}$$

Let u be any matrix in $SU(2)$: $u^{-1} = u^\dagger$ and $\det(u) = 1$. Consider the mapping:

$$X \to X' = uXu^\dagger. \tag{39}$$

We demonstrate that this is a linear mapping of V_3 into itself: If X is hermitian, so is X'. If X is traceless then so is X':

$$\mathrm{Tr}(X') = \mathrm{Tr}(uXu^\dagger) = \mathrm{Tr}(Xuu^\dagger) = \mathrm{Tr}(X). \tag{40}$$

This mapping of V_3 into itself also preserves the norms, and hence, the scalar products:

$$\begin{aligned}(X', X') &= \frac{1}{2}\mathrm{Tr}(X'X') \\ &= \frac{1}{2}\mathrm{Tr}(uXu^\dagger uXu^\dagger) \\ &= (X, X).\end{aligned} \tag{41}$$

The mapping is therefore a rotation acting on V_3 and we find that to every element of $SU(2)$ there corresponds a rotation.

Let us make this notion of a connection more explicit, by picking an orthonormal basis $(\sigma_1, \sigma_2, \sigma_3)$ of V_3, in the form of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{42}$$

Note that the Pauli matrices form an orthonormal basis:

$$\frac{1}{2}\mathrm{Tr}(\sigma_\alpha \sigma_\beta) = \delta_{\alpha\beta}. \tag{43}$$

We have the products:

$$\sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_3 = i\sigma_1, \quad \sigma_3\sigma_1 = i\sigma_2. \tag{44}$$

Different Pauli matrices anti-commute:

$$\{\sigma_\alpha, \sigma_\beta\} \equiv \sigma_\alpha\sigma_\beta + \sigma_\beta\sigma_\alpha = 2\delta_{\alpha\beta} I \tag{45}$$

The commutation relations are:

$$[\sigma_\alpha, \sigma_\beta] = 2i\epsilon_{\alpha\beta\gamma}\sigma_\gamma. \tag{46}$$

Any element of V_3 may be written in the form:

$$X = \mathbf{x} \cdot \boldsymbol{\sigma} = \sum_{i=1}^{3} x_i \sigma_i, \tag{47}$$

where $\mathbf{x} \in R^3$. This establishes a $1:1$ correspondence between elements of V_3 and R^3. We note that

$$\frac{1}{2}\mathrm{Tr}\left[(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma})\right] = \mathbf{a}\cdot\mathbf{b}, \tag{48}$$

and

$$\begin{aligned}(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) &= \sum_{i=1}^{3} a_i b_i \sigma_i^2 + \sum_{i=1}^{3}\sum_{j\neq i} a_i b_j \sigma_i \sigma_j \\ &= (\mathbf{a}\cdot\mathbf{b})I + i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}. \end{aligned} \tag{49}$$

Finally, we may see that the mapping is isometric:

$$(X,Y) = \frac{1}{2}\mathrm{Tr}(XY) = \frac{1}{2}\mathrm{Tr}\left[(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{y}\cdot\boldsymbol{\sigma})\right] = \mathbf{x}\cdot\mathbf{y}. \tag{50}$$

Let's investigate $SU(2)$ further, and see the relevance of V_3: Every unitary matrix can be expressed as the exponential of a skew-hermitian ($A^\dagger = -A$) matrix. If the unitary matrix is also unimodular, then the skew-hermitian matrix can be selected to be traceless. Hence, every $u \in SU(2)$ is of the form $u = e^{-iH}$, where $H = H^\dagger$ and $\mathrm{Tr}(H) = 0$. For every real unit vector $\mathbf{e}$ and every real θ, we define $u_{\mathbf{e}}(\theta) \in SU(2)$ by:

$$u(\theta) = \exp\left(-\frac{i}{2}\theta\mathbf{e}\cdot\boldsymbol{\sigma}\right). \tag{51}$$

Any element of $SU(2)$ can be expressed in this form, since every traceless hermitian matrix is a (real) linear combination of the Pauli matrices.

Now let us relate $u_{\mathbf{e}}(\theta)$ to the rotation $R_{\mathbf{e}}(\theta)$:

Theorem: Let $\mathbf{x} \in R^3$, and $X = \mathbf{x}\cdot\boldsymbol{\sigma}$. Let

$$u(\theta) = \exp\left(-\frac{i}{2}\theta\mathbf{e}\cdot\boldsymbol{\sigma}\right), \tag{52}$$

and let

$$u_{\mathbf{e}}(\theta) X u_{\mathbf{e}}^\dagger(\theta) = X' = \mathbf{x}'\cdot\boldsymbol{\sigma}. \tag{53}$$

Then

$$\mathbf{x}' = R_{\mathbf{e}}(\theta)\mathbf{x}. \tag{54}$$

Proof: Note that

$$u_{\mathbf{e}}(\theta) = \exp\left(-\frac{i}{2}\theta\mathbf{e}\cdot\boldsymbol{\sigma}\right) = I\cos\frac{\theta}{2} - i(\mathbf{e}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2}. \tag{55}$$

This may be demonstrated by using the identity $(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\mathbf{b})I + (\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}$, and letting $\mathbf{a}=\mathbf{b}=\mathbf{e}$ to get $(\mathbf{e}\cdot\sigma)^2 = I$, and using this to sum the exponential series.

Thus,

$$\begin{aligned}
\mathbf{x}'\cdot\boldsymbol{\sigma} &= u_{\mathbf{e}}(\theta)Xu^{\dagger}_{\mathbf{e}}(\theta)\\
&= \left[I\cos\frac{\theta}{2} - i(\mathbf{e}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2}\right](\mathbf{x}\cdot\boldsymbol{\sigma})\left[I\cos\frac{\theta}{2} + i(\mathbf{e}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2}\right]\\
&= \mathbf{x}\cdot\sigma\cos^2\frac{\theta}{2} + (\mathbf{e}\cdot\boldsymbol{\sigma})(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{e}\cdot\boldsymbol{\sigma})\sin^2\frac{\theta}{2}\\
&\quad + [-i(\mathbf{e}\cdot\boldsymbol{\sigma})(\mathbf{x}\cdot\boldsymbol{\sigma}) + i(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{e}\cdot\boldsymbol{\sigma})]\sin\frac{\theta}{2}\cos\frac{\theta}{2}.
\end{aligned} \tag{56}$$

But

$$\begin{aligned}
(\mathbf{e}\cdot\boldsymbol{\sigma})(\mathbf{x}\cdot\boldsymbol{\sigma}) &= (\mathbf{e}\cdot\mathbf{x})I + i(\mathbf{e}\times\mathbf{x})\cdot\boldsymbol{\sigma} && (57)\\
(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{e}\cdot\boldsymbol{\sigma}) &= (\mathbf{e}\cdot\mathbf{x})I - i(\mathbf{e}\times\mathbf{x})\cdot\boldsymbol{\sigma} && (58)\\
(\mathbf{e}\cdot\boldsymbol{\sigma})(\mathbf{x}\cdot\boldsymbol{\sigma})(\mathbf{e}\cdot\boldsymbol{\sigma}) &= (\mathbf{e}\cdot\mathbf{x})(\mathbf{e}\cdot\sigma) + i\left[(\mathbf{e}\times\mathbf{x})\cdot\boldsymbol{\sigma}\right](\mathbf{e}\cdot\boldsymbol{\sigma})\\
&= (\mathbf{e}\cdot\mathbf{x})(\mathbf{e}\cdot\boldsymbol{\sigma}) + i^2\left[(\mathbf{e}\times\mathbf{x})\times\mathbf{e}\right]\cdot\boldsymbol{\sigma}\\
&= 2(\mathbf{e}\cdot\mathbf{x})(\mathbf{e}\cdot\boldsymbol{\sigma}) - \mathbf{x}\cdot\boldsymbol{\sigma}, && (59)
\end{aligned}$$

where we have made use of the identity $(\mathbf{C}\times\mathbf{B})\times\mathbf{A} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ to obtain $(\mathbf{e}\times\mathbf{x})\times\mathbf{e} = \mathbf{x} - \mathbf{e}(\mathbf{e}\cdot\mathbf{x})$. Hence,

$$\begin{aligned}
\mathbf{x}'\cdot\boldsymbol{\sigma} &= \left\{\cos^2\frac{\theta}{2}\mathbf{x} + \sin^2\frac{\theta}{2}\left[2(\mathbf{e}\cdot\mathbf{x})\mathbf{e} - \mathbf{x}\right]\right\}\cdot\boldsymbol{\sigma}\\
&\quad + i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left[-2i(\mathbf{e}\times\mathbf{x})\cdot\boldsymbol{\sigma}\right].
\end{aligned} \tag{60}$$

Equating coefficients of σ we obtain:

$$\begin{aligned}
\mathbf{x}' &= \mathbf{x}\cos^2\frac{\theta}{2} + [2(\mathbf{e}\cdot\mathbf{x})\mathbf{e} - \mathbf{x}]\sin^2\frac{\theta}{2} + 2(\mathbf{e}\times\mathbf{x})\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\
&= (\mathbf{e}\cdot\mathbf{x})\mathbf{e} + [\mathbf{x} - (\mathbf{e}\cdot\mathbf{x})\mathbf{e}]\cos\theta + (\mathbf{e}\times\mathbf{x})\sin\theta\\
&= R_{\mathbf{e}}(\theta)\mathbf{x}.
\end{aligned} \tag{61}$$

Thus, we have shown that to every rotation $R_{\mathbf{e}}(\theta)$ corresponds at least one $u \in SU(2)$, and also to every element of $SU(2)$ there corresponds a rotation. We may restate the theorem just proved in the alternative form:

$$\begin{aligned} uXu^\dagger &= u(\mathbf{x}\cdot\boldsymbol{\sigma})u^\dagger = \mathbf{x}\cdot(u\boldsymbol{\sigma}u^\dagger) \\ &= \mathbf{x}\cdot\boldsymbol{\sigma} = [R_{\mathbf{e}}(\theta)\mathbf{x}]\cdot\sigma = \mathbf{x}\cdot\left[R_{\mathbf{e}}^{-1}(\theta)\boldsymbol{\sigma}\right]. \end{aligned} \tag{62}$$

But $\mathbf{x}$ is arbitrary, so

$$u\boldsymbol{\sigma}u^\dagger = R_{\mathbf{e}}^{-1}(\theta)\boldsymbol{\sigma}, \tag{63}$$

or,

$$u^{-1}\boldsymbol{\sigma}u = R_{\mathbf{e}}(\theta)\boldsymbol{\sigma}. \tag{64}$$

More explicitly, this means:

$$u^{-1}\sigma_i u = \sum_{j=1}^{3} R_{\mathbf{e}}(\theta)_{ij}\sigma_j. \tag{65}$$

There remains the question of uniqueness: Suppose $u_1 \in SU(2)$ and $u_2 \in SU(2)$ are such that

$$u_1 X u_1^\dagger = u_2 X u_2^\dagger, \quad \forall X \in V_3. \tag{66}$$

Then $u_2^{-1}u_1$ commutes with every $X \in V_3$ and therefore this matrix must be a multiple of the identity (left for the reader to prove). Since it is unitary and unimodular, it must equal I or $-I$. Thus, there is a two-to-one correspondence between $SU(2)$ and $O^+(3)$: To every rotation $R_{\mathbf{e}}(\theta)$ corresponds the pair $u_{\mathbf{e}}(\theta)$ and $-u_{\mathbf{e}}(\theta) = u_{\mathbf{e}}(\theta+2\pi)$. Such a mapping of $SU(2)$ onto $O^+(3)$ is called a **homomorphism** (alternatively called an **unfaithful representation**).

We make this correspondence precise in the following:

Theorem: 1. There is a two-to-one correspondence between $SU(2)$ and $O^+(3)$ under the mapping:

$$u \to R(u), \quad \text{where} \quad R_{ij}(u) = \frac{1}{2}\mathrm{Tr}(u^\dagger\sigma_i u\sigma_j), \tag{67}$$

and the rotation $R_{\mathbf{e}}(\theta)$ corresponds to the pair:

$$R_{\mathbf{e}}(\theta) \leftrightarrow \{u_{\mathbf{e}}(\theta), -u_{\mathbf{e}}(\theta) = u_{\mathbf{e}}(\theta+2\pi)\}. \tag{68}$$

2. In particular, the pair of elements $\{I, -I\} \subset SU(2)$ maps to $I \in O^+(3)$.
3. This mapping is a homomorphism: $u \to R(u)$ is a **representation** of $SU(2)$, such that

$$R(u'u'') = R(u')R(u''), \quad \forall u', u'' \in SU(2). \tag{69}$$

That is, the "multiplication table" is preserved under the mapping.

Proof: 1. We have

$$u^{-1}\sigma_i u = \sum_{j=1}^{3} R_{\mathbf{e}}(\theta)_{ij}\sigma_j. \tag{70}$$

Multiply by σ_k and take the trace:

$$\mathrm{Tr}(u^{-1}\sigma_i u\sigma_k) = \mathrm{Tr}\left[\sum_{j=1}^{3} R_{\mathbf{e}}(\theta)_{ij}\sigma_j\sigma_k\right], \tag{71}$$

or

$$\mathrm{Tr}(u^\dagger\sigma_i u\sigma_k) = \sum_{j=1}^{3} R_{\mathbf{e}}(\theta)_{ij}\mathrm{Tr}(\sigma_j\sigma_k). \tag{72}$$

But $\frac{1}{2}\mathrm{Tr}(\sigma_j\sigma_k) = \delta_{jk}$, hence

$$R_{ik}(u) = R_{\mathbf{e}}(\theta)_{ik} = \frac{1}{2}\mathrm{Tr}(u^\dagger\sigma_i u\sigma_k). \tag{73}$$

Proof of the remaining statements is left to the reader.

A couple of comments may be helpful here:

1. Why did we restrict u to be unimodular? That is, why are we considering $SU(2)$, and not $U(2)$. In fact, we could have considered $U(2)$, but the larger group only adds unnecessary complication. All $U(2)$ adds is multiplication by an overall phase factor, and this has no effect in the transformation:

$$X \to X' = uXu^\dagger. \tag{74}$$

This would enlarge the two-to-one mapping to infinity-to-one, apparently without achieving anything of interest. So, we keep things as simple as we can make them.

2. Having said that, can we make things even simpler? That is, can we impose additional restrictions to eliminate the "double-valuedness" in the above theorem? The answer is no – $SU(2)$ has no subgroup which is isomorphic with O^+3.

5 Lie Groups: $O^+(3)$ and $SU(2)$

Def: An abstract n-dimensional **Lie algebra** is an n-dimensional vector space $\mathcal{V}$ on which is defined the notion of a product of two vectors $(*)$ with the properties ($x, y, z \in \mathcal{V}$, c a complex number):

1. Closure: $x * y \in \mathcal{V}$.
2. Distributivity:

$$\begin{aligned} x * (y + z) &= x * y + x * z && (75) \\ (y + z) * x &= y * x + z * x. && (76) \end{aligned}$$

3. Associativity with respect to multiplication by a complex number:

$$(cx) * y = c(x * y). \quad (77)$$

4. Anti-commutativity:

$$x * y = -y * x \quad (78)$$

5. Non-associative ("Jacobi identity"):

$$x * (y * z) + z * (x * y) + y * (z * x) = 0 \quad (79)$$

We are especially interested here in Lie algebras realized in terms of matrices (in fact, every finite-dimensional Lie algebra has a faithful representation in terms of finite-dimensional matrices):

Def: A Lie algebra of matrices is a vector space $\mathcal{M}$ of matrices which is closed under the operation of forming the commutator:

$$[M', M''] = M'M'' - M''M' \in \mathcal{M}, \quad \forall M', M'' \in \mathcal{M}. \quad (80)$$

Thus, the Lie product is the commutator: $M' * M'' = [M', M'']$. The vector space may be over the real or complex fields.

Let's look at a couple of relevant examples:

1. The set of all real skew-symmetric 3×3 matrices is a three-dimensional Lie algebra. Any such matrix is a real linear combination of the matrices

$$\mathcal{J}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{81}$$

   as defined already earlier. The basis vectors satisfy the commutation relations:

$$[\mathcal{J}_i, \mathcal{J}_j] = \epsilon_{ijk}\mathcal{J}_k. \tag{82}$$

   We say that this Lie algebra is the Lie algebra associated with the group $O^+(3)$. Recall that

$$R_{\mathbf{u}}(\theta) = e^{\theta\mathbf{u}\cdot\boldsymbol{\mathcal{J}}}. \tag{83}$$

2. The set of all 2×2 skew-hermitian matrices is a Lie algebra of matrices. This is also 3-dimensional, and if we write:

$$\mathcal{S}_j = \frac{i}{2}\sigma_j, \quad j = 1, 2, 3, \tag{84}$$

   we find $\{\mathcal{S}\}$ satisfy the "same" commutation relations as $\{\mathcal{J}\}$:

$$[\mathcal{S}_i, \mathcal{S}_j] = \epsilon_{ijk}\mathcal{S}_k. \tag{85}$$

   This is the Lie algebra associated with the group $SU(2)$. Recall that

$$u_{\mathbf{e}}(\theta) = e^{\theta\mathbf{e}\cdot(-\frac{i}{2}\boldsymbol{\sigma})} = e^{\theta\mathbf{e}\cdot\boldsymbol{\mathcal{S}}}. \tag{86}$$

   This is also a real Lie algebra, *i.e.*, a vector space over the real field, even though the matrices are not in general real.

We see that the Lie algebras of $O^+(3)$ and $SU(2)$ have the same "structure", *i.e.*, a $1:1$ correspondence can be established between them which is linear and preserves all commutators. As Lie algrebras, the two are isomorphic.

We explore a bit more the connection between Lie algebras and Lie groups. Let $\mathcal{M}$ be an n-dimensional Lie algebra of matrices. Associated with $\mathcal{M}$ there is an n-dimensional **Lie group** $\mathcal{G}$ of matrices: $\mathcal{G}$ is the matrix group **generated** by all matrices of the form e^X, where $X \in \mathcal{M}$. We see that $O^+(3)$ and $SU(2)$ are Lie groups of this kind – in fact, every element of either of these groups corresponds to an exponential of an element of the appropriate Lie algrebra.[1]

6 Continuity Structure

As a finite dimensional vector space, $\mathcal{M}$ has a continuity structure in the usual sense (*i.e.*, it is a topological space with the "usual" topology). This induces a continuity structure (topology) on $\mathcal{G}$ (for $O^+(3)$ and $SU(2)$, there is nothing mysterious about this, but we'll keep our discussion a bit more general for a while). $\mathcal{G}$ is an n-dimensional manifold (a topological space such that every point has a neighborhood which can be mapped homeomorphically onto n-dimensional Euclidean space). The structure of $\mathcal{G}$ (its multiplication table) in some neighborhood of the identity is uniquely determined by the structure of the Lie algebra $\mathcal{M}$. This statement follows from the Campbell-Baker-Hausdorff theorem for matrices: If matrices X, Y are sufficiently "small", then $e^X e^Y = e^Z$, where Z is a matrix in the Lie algebra generated by matrices X and Y. That is, Z is a series of repeated commutators of the matrices X and Y. Thus, we have the notion that the *local* structure of $\mathcal{G}$ is determined solely by the structure of $\mathcal{M}$ as a Lie algebra.

We saw that the Lie algebras of $O^+(3)$ and $SU(2)$ are isomorphic, hence the group $O^+(3)$ is *locally* isomorphic with $SU(2)$. Note, on the other hand, that the properties

$$(\mathbf{u}\cdot\mathcal{J})^3 = -(\mathbf{u}\cdot\mathcal{J}) \quad \text{and} \quad (\mathbf{u}\cdot\mathcal{J})^{2n+m} = (-)^n(\mathbf{u}\cdot\mathcal{J})^m, \tag{87}$$

for all positive integers n, m, are not shared by the Pauli matrices, which instead satisfy:

$$(\mathbf{u}\cdot\sigma)^3 = \mathbf{u}\cdot\sigma. \tag{88}$$

Such algebraic properties are outside the realm of Lie algebras (the products being taken are not Lie products). We also see that (as with $O^+(3)$ and

[1]This latter fact is not a general feature of Lie groups: To say that $\mathcal{G}$ is generated by matrices of the form e^X means that $\mathcal{G}$ is the intersection of all matrix groups which contain all matrices e^X where $X \in \mathcal{M}$. An element of $\mathcal{G}$ may not be of the form e^X.

$SU(2)$) it is possible for two Lie algebras to have the same local structure, while not being globally isomorphic.

A theorem describing this general situation is the following:

Theorem: (and definition) To every Lie algebra $\mathcal{M}$ corresponds a unique simply-connected Lie group, called the **Universal Covering Group**, defined by $\mathcal{M}$. Denote this group by $\mathcal{G}_U$. Every other Lie group with a Lie algebra isomorphic with $\mathcal{M}$ is then isomorphic with the quotient group of $\mathcal{G}_U$ relative to some discrete (central – all elements which map to the identity) subgroup of $\mathcal{G}_U$ iself. If the other group is simply connected, it is isomorphic with $\mathcal{G}_U$ itself.

We apply this to rotations: The group $SU(2)$ is the universal covering group defined by the Lie algebra of the rotation group, hence $SU(2)$ takes on special significance. We note that $SU(2)$ can be parameterized as the surface of a unit sphere in four dimensions, hence is simply connected (all closed loops may be continuously collapsed to a point). On the other hand, $O^+(3)$ is isomorphic with the quotient group $SU(2)/\mathcal{I}(2)$, where $\mathcal{I}(2)$ is the inversion group in two dimensions:

$$\mathcal{I}(2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{89}$$

7 The Haar Integral

We shall find it desirable to have the ability to perform an "invariant integral" on the manifolds $O^+(3)$ and $SU(2)$, which in some sense assigns an equal "weight" to every element of the group. The goal is to find a way of democratically "averaging" over the elements of a group. For a finite group, the correspondence is to a sum over the group elements, with the same weight for each element.

For the rotation group, let us denote the desired "volume element" by $d(R)$. We must find an expression for $d(R)$ in terms of the parameterization, for some parameterization of $O^+(3)$. For example, we consider the Euler angle parameterization. Recall, in terms of Euler angles the representation of a rotation as:

$$R = R(\psi, \theta, \phi) = R_{\mathbf{e}_3}(\psi) R_{\mathbf{e}_2}(\theta) R_{\mathbf{e}_3}(\phi). \tag{90}$$

We will argue that the appropriate volume element must be of the form:

$$d(R) = K d\psi \sin\theta d\theta d\phi, \quad K > 0. \tag{91}$$

The argument for this form is as follows: We have a 1 : 1 correspondence between elements of $O^+(3)$ and orientations of a rigid body (such as a sphere with a dot at the north pole, centered at the origin; let $I \in O^+(3)$ correspond to the orientation with the north pole on the $+\mathbf{e}_3$ axis, and the meridian along the $+\mathbf{e}_2$ axis, say). We want to find a way to average over all positions of the sphere, with each orientation receiving the same weight. This corresponds to a uniform averaging over the sphere of the location of the north pole.

Now notice that if $R(\psi, \theta, \phi)$ acts on the reference position, we obtain an orientation where the north pole has polar angles (θ, ψ). Thus, the (θ, ψ) dependence of $d(R)$ must be $d\psi \sin\theta d\theta$. For fixed (θ, ψ), the angle ϕ describes a rotation of the sphere about the north-south axis – the invariant integral must correspond to a uniform averaging over this angle. Hence, we intuitively arrive at the above form for $d(R)$. The constant $K > 0$ is arbitrary; we pick it for convenience. We shall choose K so that the integral over the entire group is one:

$$1 = \int_{O^+(3)} d(R) = \frac{1}{8\pi^2} \int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi. \tag{92}$$

We can thus evaluate the integral of a (suitably behaved) function $f(R) = f(\psi, \theta, \phi)$ over $O^+(3)$:

$$\overline{f(R)} = \int_{O^+(3)} f(R) d(R) = \frac{1}{8\pi^2} \int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi f(\psi, \theta, \phi). \tag{93}$$

The overbar notation is intended to suggest an average.

What about the invariant integral over $SU(2)$? Given the answer for $O^+(3)$, we can obtain the result for $SU(2)$ using the connection between the two groups. First, parameterize $SU(2)$ by the Euler angles:

$$u(\psi, \theta, \phi) = \exp\left(-\frac{i}{2}\psi\sigma_3\right) \exp\left(-\frac{i}{2}\theta\sigma_2\right) \exp\left(-\frac{i}{2}\phi\sigma_3\right), \tag{94}$$

with

$$0 \le \psi < 2\pi; \quad 0 \le \theta \le \pi; \quad 0 \le \phi < 4\pi. \tag{95}$$

Notice that the ranges are the same as for $O^+(3)$, except for the doubled range required for ϕ. With these ranges, we obtain every element of $SU(2)$, uniquely, up to a set of measure 0 (when $\theta = 0, \pi$). The integral of function $g(u)$ on $SU(2)$ is thus:

$$\overline{g(u)} = \int_{SU(2)} g(u)d(u) = \frac{1}{16\pi^2}\int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta d\theta \int_0^{4\pi} d\phi g\left[u(\psi,\theta,\phi)\right], \quad (96)$$

with the volume element normalized to give unit total volume:

$$\int_{SU(2)} d(u) = 1. \quad (97)$$

A more precise mathematical treatment is possible, making use of measure theory; we'll mention some highlights here. The goal is to define a measure $\mu(S)$ for suitable subsets of $O^+(3)$ (or $SU(2)$) such that if R_0 is any element of $O^+(3)$, then:

$$\mu(SR_0) = \mu(S), \quad \text{where } SR_0 = \{RR_0 | R \in S\}. \quad (98)$$

Intuitively, we think of the following picture: S may be some region in $O^+(3)$, and SR_0 is the image of S under the mapping $R \to RR_0$. The idea then, is that the regions S and SR_0 should have the same "volume" for all $R \in O^+(3)$. Associated with such a measure we have an integral.

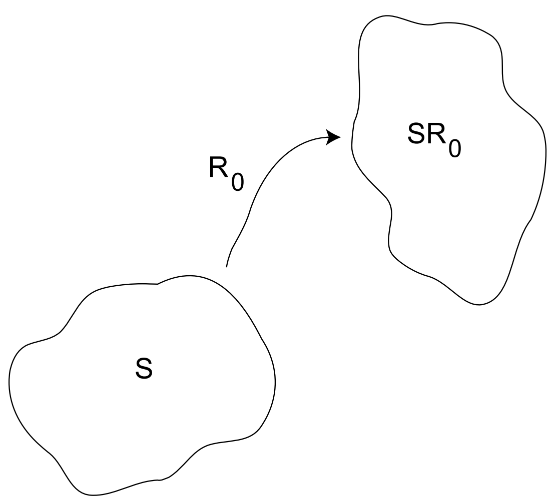

Figure 5: Set S mapping to set SR_0, under the rotation R_0.

It may be shown[2] that a measure with the desired property exists and is unique, up to a factor, for any finite-dimensional Lie group. Such a measure is called a **Haar measure**. The actual construction of such a measure must deal with coordinate system issues. For example, there may not be a good global coordinate system on the group, forcing the consideration of local coordinate systems.

Fortunately, we have already used our intuition to obtain the measure (volume element) for the Euler angle parameterization, and a rigorous treatment would show it to be correct. The volume element in other parameterizations may be found from this one by suitable Jacobian calculations. For example, if we parameterize $O^+(3)$ by:

$$R_{\mathbf{e}}(\theta) = e^{\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{J}}}, \tag{99}$$

where $\boldsymbol{\theta} \equiv \theta\mathbf{e}$, and $|\boldsymbol{\theta}| \leq \pi$, then the volume element (normalized again to unit total volume of the group) is:

$$d(R) = \frac{1}{4\pi^2}\frac{1-\cos\theta}{\theta^2}d^3(\boldsymbol{\theta}), \tag{100}$$

where $d^3(\boldsymbol{\theta})$ is an ordinary volume element on R^3. Thus, the group-averaged value of $f(R)$ is:

$$\begin{aligned} \int_{O^+(3)} f(R)d(R) &= \frac{1}{4\pi^2}\int_{O^+(3)}\frac{1-\cos\theta}{\theta^2}d^3(\boldsymbol{\theta})f\left(e^{\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{J}}}\right) & (101)\\ &= \frac{1}{4\pi^2}\int_{4\pi}d\Omega_{\mathbf{e}}\int_0^{2\pi}(1-\cos\theta)d\theta f\left(e^{\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{J}}}\right). & (102)\end{aligned}$$

Alternatively, we may substitute $1-\cos\theta = 2\sin^2\frac{\theta}{2}$. For $SU(2)$ we have the corresponding result:

$$d(u) = \frac{1}{4\pi^2}d\Omega_{\mathbf{e}}\sin^2\frac{\theta}{2}d\theta, \quad 0 \leq \theta \leq 2\pi. \tag{103}$$

We state without pro0f that the Haar measure is both left- and right-invariant. That is, $\mu(S) = \mu(SR_0) = \mu(R_0S)$ for all $R_0 \in O^+(3)$ and for all measurable sets $S \subset O^+(3)$. This is to be hoped for on "physical" grounds. The invariant integral is known as the **Haar integral**, or its particular realization for the rotation group as the **Hurwitz integral**.

[2]This may be shown by construction, starting with a small neighborhood of the identity, and using the desired property to transfer the right volume element everywhere.

8 Unitary Representations of $SU(2)$ (and $O^+(3)$)

A unitary representation of $SU(2)$ is a mapping

$$u \in SU(2) \to U(u) \in U(n) \quad \text{such that} \quad U(u')U(u'') = U(u'u''), \forall u', u'' \in SU(2). \tag{104}$$

That is, we "represent" the elements of $SU(2)$ by unitary matrices (not necessarily 2×2), such that the multiplication table is preserved, either homomorphically or isomorphically. We are very interested in such mappings, because they permit the study of systems of arbitrary angular momenta, as well as providing the framework for adding angular momenta, and for studying the angular symmetry properties of operators. We note that, for every unitary representation $R \to T(R)$ of $O^+(3)$ there corresponds a unitary representation of $SU(2)$: $u \to U(u) = T[R(u)]$. Thus, we focus our attention on $SU(2)$, without losing any generality.

For a physical theory, it seems reasonable to to demand some sort of continuity structure. That is, whenever two rotations are near each other, the representations for them must also be close.

Def: A unitary representation $U(u)$ is called **weakly continuous** if, for any two vectors ϕ, ψ, and any u:

$$\lim_{u' \to u} \langle \phi | \left[U(u') - U(u) \right] \psi \rangle = 0. \tag{105}$$

In this case, we write:

$$\operatorname*{w-lim}_{u' \to u} U(u') = U(u), \tag{106}$$

and refer to it as the "weak-limit".

Def: A unitary representation $U(u)$ is called **strongly continuous** if, for any vector ϕ and any u:

$$\lim_{u' \to u} \| \left[U(u') - U(u) \right] \phi \| = 0. \tag{107}$$

In this case, we write:

$$\operatorname*{s-lim}_{u' \to u} U(u') = U(u), \tag{108}$$

and refer to it as the "strong-limit".

Strong continuity implies weak continuity, since:

$$|\langle\phi| \left[U(u') - U(u)\right] \psi\rangle| \leq \|\phi\| \| \left[U(u') - U(u)\right] \psi\| \qquad (109)$$

We henceforth (*i.e.*, until experiment contradicts us) adopt these notions of continuity as physical requirements.

An important concept in representation theory is that of "(ir)reducibility":

Def: A unitary representation $U(u)$ is called **irreducible** if no subspace of the Hilbert space is mapped into itself by every $U(u)$. Otherwise, the representation is said to be **reducible**.

Irreducible representations are discussed so frequently that the jargon "**irrep**" has emerged as a common substitute for the somewhat lengthy "irreducible representation".

Lemma: A unitary representation $U(u)$ is irreducible if and only if every bounded operator Q which commutes with every $U(u)$ is a multiple of the identity.

Proof of this will be left as an exercise. Now for one of our key theorems:

Theorem: If $u \to U(u)$ is a strongly continuous irreducible representation of $SU(2)$ on a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ has a finite number of dimensions.

Proof: The proof consists of showing that we can place a finite upper bound on the number of mutually orthogonal vectors in $\mathcal{H}$: Let E be any one-dimensional projection operator, and ϕ, ψ be any two vectors in $\mathcal{H}$. Consider the integral:

$$B(\psi, \phi) = \int_{SU(2)} d(u) \langle\psi|U(u)EU(u^{-1})\phi\rangle. \qquad (110)$$

This integral exists, since the integrand is continuous and bounded, because $U(u)EU(u^{-1})$ is a one-dimensional projection, hence of norm 1.

Now

$$\begin{aligned} |B(\psi, \phi)| &= \left| \int_{SU(2)} d(u) \langle\psi|U(u)EU(u^{-1})\phi\rangle \right| \\ &\leq \int_{SU(2)} d(u) |\langle\psi|U(u)EU(u^{-1})\phi\rangle| \qquad (111) \\ &\leq \int_{SU(2)} d(u) \|\psi\|\|\phi\|\|U(u)EU(u^{-1})\| \qquad (112) \\ &\leq \|\psi\|\|\phi\|, \qquad (113) \end{aligned}$$

where we have made use of the Schwarz inequality and of the fact $\int_{SU(2)} d(u) = 1$.

$B(\psi, \phi)$ is linear in ϕ, anti-linear in ψ, and hence defines a bounded operator B_0 such that:

$$B(\psi, \phi) = \langle \psi | B_0 \phi \rangle. \tag{114}$$

Let $u_0 \in SU(2)$. Then

$$\begin{aligned}
\langle \psi | U(u_0) B_0 U(u_0^{-1}) \phi \rangle &= \int_{SU(2)} d(u) \langle \psi | U(u_0 u) E U((u_0 u)^{-1}) \phi \rangle && (115) \\
&= \int_{SU(2)} d(u) \langle \psi | U(u) E U(u^{-1}) \phi \rangle && (116) \\
&= \langle \psi | B_0 \phi \rangle, && (117)
\end{aligned}$$

where the second line follows from the invariance of the Haar integral. Since ψ and ϕ are arbitrary vectors, we thus have;

$$U(u_0) B_0 = B_0 U(u_0), \quad \forall u_0 \in SU(2). \tag{118}$$

Since B_0 commutes with every element of an irreducible representation, it must be a multiple of the identity, $B_0 = pI$.

$$\int_{SU(2)} d(u) U(u) E U(u^{-1}) = pI. \tag{119}$$

Now let $\{\phi_n | n = 1, 2, \ldots, N\}$ be a set of N orthonormal vectors, $\langle \phi_n | \phi_m \rangle = \delta_{nm}$, and take $E = |\phi_1\rangle\langle\phi_1|$. Then,

$$\begin{aligned}
\langle \phi_1 | \int_{SU(2)} d(u) U(u) E U(u^{-1}) | \phi_1 \rangle &= p \langle \phi_1 | I | \phi_1 \rangle = p, && (120) \\
&= \int_{SU(2)} d(u) \langle \phi_1 | U(u) | \phi_1 \rangle \langle \phi_1 | U(u^{-1}) | \phi_1 \rangle \\
&= \int_{SU(2)} d(u) |\langle \phi_1 | U(u) | \phi_1 \rangle|^2 > 0. && (121)
\end{aligned}$$

Note that the integral cannot be zero, since the integrand is a continuous non-negative definite function of u, and is equal to one for $u = I$.

Thus, we have:

$$\begin{aligned}
pN &= \sum_{n=1}^{N} \langle\phi_n|pI\phi_n\rangle && (122)\\
&= \sum_{n=1}^{N} \int_{SU(2)} d(u)\langle\phi_n|U(u)EU(u^{-1})\phi_n\rangle && (123)\\
&= \sum_{n=1}^{N} \int_{SU(2)} d(u)\langle\phi_n|U(u)|\phi_1\rangle\langle\phi_1|U(u^{-1})\phi_n\rangle && (124)\\
&= \sum_{n=1}^{N} \int_{SU(2)} d(u)\langle U(u)\phi_1|\phi_n\rangle\langle\phi_n|U(u)\phi_1\rangle && (125)\\
&= \int_{SU(2)} d(u)\langle U(u)\phi_1|\sum_{n=1}^{N} |\phi_n\rangle\langle\phi_n|U(u)\phi_1\rangle && (126)\\
&\leq \int_{SU(2)} d(u)\langle U(u)\phi_1|I|U(u)\phi_1\rangle && (127)\\
&\leq \int_{SU(2)} d(u)\|U(u)\phi_1\|^2 = 1. && (128)\\
& && (129)
\end{aligned}$$

That is, $pN \leq 1$. But $p > 0$, so $N < \infty$, and hence $\mathcal{H}$ cannot contain an arbitrarily large number of mutually orthogonal vectors. In other words, $\mathcal{H}$ is finite-dimensional.

Thus, we have the important result that if $U(u)$ is irreducible, then the operators $U(u)$ are finite-dimensional unitary matrices. We will not have to worry about delicate issues that might arise if the situation were otherwise.[3]

Before actually building representations, we would like to know whether it is "sufficient" to consider only unitary representations of $SU(2)$.

Def: Two (finite dimensional) representations U and W of a group are called **equivalent** if and only if they are similar, that is, if there exists a fixed similarity transformation S which maps one representation onto the other:

$$U(u) = SW(u)S^{-1}, \quad \forall u \in SU(2). \tag{130}$$

Otherwise, the representations are said to be **inequivalent**.

[3] The theorem actually holds for any compact Lie group, since a Haar integral normalized to one exists.

Note that we can think of equivalence as just a basis transformation. The desired theorem is:

Theorem: Any finite-dimensional (continuous) representation $u \to W(u)$ of $SU(2)$ is equivalent to a (continuous) unitary representation $u \to U(u)$ of $SU(2)$.

Proof: We prove this theorem by constructing the required similarity transformation: Define matrix

$$P = \int_{SU(2)} d(u) W^\dagger(u) W(u). \tag{131}$$

This matrix is positive definite and Hermitian, since the integrand is. Thus P has a unique positive-definite Hermitian square root S:

$$P = P^\dagger > 0 \Longrightarrow \sqrt{P} = S = S^\dagger > 0. \tag{132}$$

Now, let $u_0, u \in SU(2)$. We have,

$$W^\dagger(u_0) W^\dagger(u) W(u) = \left[W^\dagger(uu_0) W(uu_0)\right] W(u_0^{-1}). \tag{133}$$

From the invariance of the Haar integral, we find:

$$\begin{aligned} W^\dagger(u_0) P &= \int_{SU(2)} d(u) \left[W^\dagger(uu_0) W(uu_0)\right] W(u_0^{-1}) && (134) \\ &= P W(u_0^{-1}), \quad \forall u_0 \in SU(2). && (135) \end{aligned}$$

Now define, for all $u \in SU(2)$,

$$U(u) = S W(u) S^{-1}. \tag{136}$$

The mapping $u \to U(u)$ defines a continuous representation of $SU(2)$, and furthermore:

$$\begin{aligned} U^\dagger(u) U(u) &= \left[S W(u) S^{-1}\right]^\dagger \left[S W(u) S^{-1}\right] \\ &= \left(S^{-1}\right)^\dagger W^\dagger(u) S^\dagger S W(u) S^{-1} \\ &= \left(S^{-1}\right)^\dagger P W^\dagger(u^{-1}) W(u) S^{-1} \\ &= \left(S^{-1}\right)^\dagger P S^{-1} \\ &= \left(S^{-1}\right)^\dagger S^\dagger S S^{-1} \\ &= I. && (137) \end{aligned}$$

That is, $U(u)$ is a unitary representation, equivalent to $W(u)$.

We have laid the fundamental groundwork: It is sufficient to determine all unitary finite-dimensional irreducible representations of $SU(2)$.

This brings us to some important "tool theorems" for working in group representaion theory.

Theorem: Let $u \to D'(u)$ and $u \to D''(u)$ be two inequivalent irreducible representations of $SU(2)$. Then the matrix elements of $D'(u)$ and $D''(u)$ satisfy:

$$\int_{SU(2)} d(u) D'_{mn}(u) D''_{rs}(u) = 0. \tag{138}$$

Proof: Note that the theorem can be thought of as a sort of orthogonality property between matrix elements of inequivalent representations. Let V' be the N'-dimensional carrier space of the representation $D'(u)$, and let V'' be the N''-dimensional carrier space of the representation $D''(u)$. Let A be any $N' \times N''$ matrix. Define another $N' \times N''$ matrix, A_0 by:

$$A_0 \equiv \int_{SU(2)} d(u) D'(u^{-1}) A D''(u). \tag{139}$$

Consider (in the sceond line, we use the invariance of the Haar integral under the substitution $u \to uu_0$):

$$\begin{aligned} D'(u_0)A_0 &= \int_{SU(2)} d(u) D'(u_0 u^{-1}) A D''(u) \\ &= \int_{SU(2)} d(u) D'(u^{-1}) A D''(uu_0) \\ &= \int_{SU(2)} d(u) D'(u^{-1}) A D''(u) D''(u_0) \\ &= A_0 D''(u_0), \quad \forall u_0 \in SU(2). \end{aligned} \tag{140}$$

Now define $N' \times N'$ matrix B' and $N'' \times N''$ matrix B'' by:

$$B' \equiv A_0 A_0^\dagger, \qquad B'' \equiv A_0^\dagger A_0. \tag{141}$$

Then we have:

$$\begin{aligned} D'(u)B' &= D'(u) A_0 A_0^\dagger \\ &= A_0 D''(u) A_0^\dagger \\ &= A_0 A_0^\dagger D'(u) \\ &= B' D'(u), \quad \forall u \in SU(2). \end{aligned} \tag{142}$$

Similarly,

$$D''(u)B'' = B''D''(u), \quad \forall u \in SU(2). \tag{143}$$

Thus, B', an operator on V', commutes with all elements of irreducible representation D', and is therefore a multiple of the identity operator on V': $B' = b'I'$. Likewise, $B'' = b''I''$ on V''.

If $A_0 \neq 0$, this can be possible only if $N' = N''$, and A_0 is non-singular. But if A_0 is non-singular, then D' and D'' are equivalent, since $D'(u)A_0 = A_0 D''$, $\forall u \in SU(2)$. But this contradicts the assumption in the theorem, hence $A_0 = 0$. To complete the proof, select $A_{nr} = 1$ for any desired n, r and set all of the other elements equal to zero.

Next, we quote the corresponding "orthonormality" theorem among elements of the same irreducible representation:

Theorem: Let $u \to D(u)$ be a (continuous) irreducible representation of $SU(2)$ on a carrier space of dimension d. Then

$$\int_{SU(2)} d(u) D_{mn}(u^{-1}) D_{rs}(u) = \delta_{ms}\delta_{nr}/d. \tag{144}$$

Proof: The proof of this theorem is similar to the preceding theorem. Let A be an arbitrary $d \times d$ matrix, and define

$$A_0 \equiv \int_{SU(2)} d(u) D(u^{-1}) A D(u). \tag{145}$$

As before, we may show that

$$D(u)A_0 = A_0 D(u), \tag{146}$$

and hence $A_0 = aI$ is a multiple of the identity. We take the trace to find the multiple:

$$a = \frac{1}{d}\mathrm{Tr}\left[\int_{SU(2)} d(u) D(u^{-1}) A D(u)\right] \tag{147}$$

$$= \frac{1}{d}\int_{SU(2)} d(u) \mathrm{Tr}\left[D(u^{-1}) A D(u)\right] \tag{148}$$

$$= \frac{1}{d}\mathrm{Tr} A. \tag{149}$$

This yields the result

$$\int_{SU(2)} d(u)D(u^{-1})AD(u) = \frac{\mathrm{Tr}(A)}{d}I. \tag{150}$$

Again, select A with any desired element equal to one, and all other elements equal to 0, to finish the proof.

We consider now the set of all irreducible representations of $SU(2)$. More precisely, we do not distinguish between equivalent representations, so this set is the union of all equivalence classes of irreducible representations. Use the symbol j to label an equivalence class, *i.e.*, j is an index, taking on values in an index set in 1:1 correspondence with the set of all equivalence classes. We denote $D(u) = D^j(u)$ to indicate that a particular irreducible representation $u \to D(u)$ belongs to equivalence class "j". Two representations $D^j(u)$ and $D^{j'}(u)$ are inequivalent if $j \neq j'$. Let d_j be the dimension associated with equivalence class j. With this new notation, we may restate our above two theorems in the form:

$$\int_{SU(2)} d(u)D^j_{mn}(u^{-1})D^{j'}_{rs} = \frac{1}{d}\delta_{jj'}\delta_{ms}\delta_{nr}. \tag{151}$$

This is an important theorem in representation theory, and is sometimes referred to as the "General Orthogonality Relation".

For much of what we need, we can deal with simpler objects than the full representation matrices. In particular, the traces are very useful invariants under similarity transformations. So, we define:

Def: The **character** $\chi(u)$ of a finite-dimensional representation $u \to D(u)$ of $SU(2)$ is the function on $SU(2)$:

$$\chi(u) = \mathrm{Tr}\left[D(u)\right]. \tag{152}$$

We immediately remark that the characters of two equivalent representations are identical, since

$$\mathrm{Tr}\left[SD(u)S^{-1}\right] = \mathrm{Tr}\left[D(u)\right]. \tag{153}$$

In fact, we shall see that the representation is completely determined by the characters, up to similarity transformations.

Let $\chi_j(u)$ denote the character of irreducible representation $D^j(u)$. The index j uniquely determines $\chi_j(u)$. We may summarize some important properties of characters in a theorem:

Theorem: 1. For any finite-dimensional representation $u \to D(u)$ of $SU(2)$:

$$\chi(u_0 u u_0^{-1}) = \chi(u), \quad \forall u, u_0 \in SU(2). \tag{154}$$

2.

$$\chi(u) = \chi^*(u) = \chi(u^{-1}) = \chi(u^*), \quad \forall u \in SU(2). \tag{155}$$

3. For the irreducible representations $u \to D^j(u)$ of $SU(2)$:

$$\int_{SU(2)} d(u)\chi_j(u_0 u^{-1}) D^{j'}(u) = \frac{1}{d_j}\delta_{jj'} D^j(u_0) \tag{156}$$

$$\int_{SU(2)} d(u)\chi_j(u_0 u^{-1}) \chi_{j'}(u) = \frac{1}{d_j}\delta_{jj'} \chi_j(u_0) \tag{157}$$

$$\int_{SU(2)} d(u)\chi_j(u^{-1})\chi_{j'}(u) = \int_{SU(2)} d(u)\chi_j^*(u)\chi_{j'}(u) = \delta_{jj'}. \tag{158}$$

Proof: (Selected portions)

1.

$$\begin{aligned} \chi(u_0 u u_0^{-1}) &= \mathrm{Tr}\left[D(u_0 u u_0^{-1})\right] \\ &= \mathrm{Tr}\left[D(u_0)D(u)D(u_0^{-1})\right] \\ &= \chi(u). \end{aligned} \tag{159}$$

2.

$$\begin{aligned} \chi(u^{-1}) &= \mathrm{Tr}\left[D(u^{-1})\right] = \mathrm{Tr}\left[D(u)^{-1}\right] \\ &= \mathrm{Tr}\left[(SU(u)S^{-1})^{-1}\right], \quad \text{where } U \text{ is a unitary representation,} \\ &= \mathrm{Tr}\left[S^{-1}U^\dagger(u)S\right] \\ &= \mathrm{Tr}\left[U^\dagger(u)\right] \\ &= \chi^*(u). \end{aligned} \tag{160}$$

The property $\chi(u) = \chi(u^*)$ holds for $SU(2)$, but not more generally [*e.g.*, it doesn't hold for $SU(3)$]. It holds for $SU(2)$ because the replacement $u \to u^*$ gives an equivalent representation for $SU(2)$. Let us demonstrate this. Consider the parameterization:

$$u = u_{\mathbf{e}}(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\mathbf{e}\cdot\boldsymbol{\sigma}. \tag{161}$$

Now form the complex conjugate, and make the following similarity transformation:

$$\begin{aligned}
\sigma_2 u^* \sigma_2^{-1} &= \sigma_2 \left[\cos\frac{\theta}{2} I + i \sin\frac{\theta}{2} \mathbf{e}\cdot\boldsymbol{\sigma}^* \sigma_2^{-1}\right] \\
&= \cos\frac{\theta}{2} I + i\sin\frac{\theta}{2}\left[e_1\sigma_2\sigma_1\sigma_2 - e_2\sigma_2\sigma_2\sigma_2 + e_3\sigma_2\sigma_3\sigma_2\right] \\
&= \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\mathbf{e}\cdot\boldsymbol{\sigma} \\
&= u. \qquad (162)
\end{aligned}$$

We thus see that u and u^* are equivalent representations for $SU(2)$. Now, for representation D (noting that $i\sigma_2 \in SU(2)$):

$$D(u) = D(i\sigma_2)D(u^*)D^{-1}(i\sigma_2), \qquad (163)$$

and hence, $\chi(u) = \chi(u^*)$.

3. We start with the general orthogonality relation, and use

$$\chi_j(u_0 u^{-1}) = \sum_{m,n} D^j_{nm}(u_0) D^j_{mn}(u^{-1}), \qquad (164)$$

to obtain

$$\begin{aligned}
\int_{SU(2)} d(u)\chi_j(u_0u^{-1})D^{j'}_{rs}(u) &= \int_{SU(2)} d(u)\sum_{m,n} D^j_{nm}(u_0)D^j_{mn}(u^{-1})D^{j'}_{rs}(u) \\
&= \sum_{m,n} D^j_{nm}(u_0)\frac{1}{d_j}\delta_{jj'}\delta_{nr}\delta_{ms} \\
&= \frac{1}{d_j}\delta_{jj'}D^j_{rs}(u_0). \qquad (165)
\end{aligned}$$

Now take the trace of both sides of this, as a matrix equation:

$$\int_{SU(2)} d(u)\chi_j(u_0u^{-1})\chi_{j'}(u) = \frac{1}{d_j}\delta_{jj'}\chi_j(u_0). \qquad (166)$$

Let $u_0 = I$. The character of the identity is just d_j, hence we obtain our last two relations.

9 Reduction of Representations

Theorem: Every continuous finite dimensional representation $u \to D(u)$ of $SU(2)$ is **completely reducible**, *i.e.*, it is the direct sum:

$$D(u) = \sum_r \oplus D^r(u) \tag{167}$$

of a finite number of irreps $D^r(u)$. The **multiplicities** m_j (the number of irreps D^r which belong to the equivalence class of irreps characterized by index j) are unique, and they are given by:

$$m_j = \int_{SU(2)} d(u)\chi_j(u^{-1})\chi(u), \tag{168}$$

where $\chi(u) = \mathrm{Tr}\,[D(u)]$. Two continuous finite dimensional representations $D'(u)$ and $D''(u)$ are equivalent if and only if their characters $\chi'(u)$ and $\chi''(u)$ are identical as functions on $SU(2)$.

Proof: It is sufficient to consider the case where $D(u)$ is unitary and reducible. In this case, there exists a proper subspace of the carrier space of $D(u)$, with projection E', which is mapped into itself by $D(u)$:

$$E'D(u)E' = D(u)E'. \tag{169}$$

Take the hermitian conjugate, and relabel $u \to u^{-1}$:

$$\begin{aligned} \left[E'D(u^{-1})E'\right]^\dagger &= \left[D(u^{-1})E'\right]^\dagger & (170)\\ E'D^\dagger(u^{-1})E' &= E'D^\dagger(u^{-1}) & (171)\\ E'D(u)E' &= E'D(u). & (172) \end{aligned}$$

Hence, $E'D(u) = D(u)E'$, for all elements u in $SU(2)$.

Now, let $E'' = I - E'$ be the projection onto the subspace orthogonal to E'. Then:

$$\begin{aligned} D(u) &= (E' + E'')D(u)(E' + E'') & (173)\\ &= E'D(u)E' + E''D(u)E'' & (174) \end{aligned}$$

(since, *e.g.*, $E'D(u)E'' = D(u)E'E'' = D(u)E'(I - E') = 0$). This formula describes a reduction of $D(u)$. If $D(u)$ restricted to subspace

E' (or E'') is reducible, we repeat the process until we have only irreps remaining. The finite dimensionality of the carrier space of $D(u)$ implies that there are a finite number of steps to this procedure.

Thus, we obtain a set of projections E_r such that

$$\begin{aligned} I &= \sum_r E_r & (175)\\ E_r E_s &= \delta_{rs} E_r & (176)\\ D(u) &= \sum_r E_r D(u) E_r, & (177) \end{aligned}$$

where $D(u)$ restricted to any of subspaces E_r is irreducible:

$$D(u) = \sum_r \oplus D^r(u). \tag{178}$$

The multiplicity follows from

$$\int_{SU(2)} d(u)\chi_j(u^{-1})\chi_{j'}(u) = \delta_{jj'}. \tag{179}$$

Thus,

$$\begin{aligned} \int_{SU(2)} d(u)\chi_j(u^{-1})\chi(u) &= \int_{SU(2)} \chi_j(u^{-1})\mathrm{Tr}\left[\sum_r \oplus D^r(u)\right] & (180)\\ &= \text{the number of terms in the sum} \\ &\quad\ \text{with } D^r = D^j & (181)\\ &= m_j. & (182) \end{aligned}$$

Finally, suppose $D'(u)$ and D'' are equivalent. We have already shown that the characters must be identical. Suppose, on the other hand, that $D'(u)$ and D'' are inequivalent. In this case, the characters cannot be identical, or this would violate our other relations above, as the reader is invited to demonstrate.

10 The Clebsch-Gordan Series

We are ready to embark on solving the problem of the addition of angular momenta. Let $D'(u)$ be a representation of $SU(2)$ on carrier space V', and let $D''(u)$ be a representation of $SU(2)$ on carrier space V'', and assume V' and V'' are finite dimensional. Let $V = V' \otimes V''$ denote the tensor product of V' and V''.

The representations D' and D'' induce a representation on V in a natural way. Define the representation $D(u)$ on V in terms of its action on any $\phi \in V$ of the form $\phi = \phi' \otimes \phi''$ as follows:

$$D(u)(\phi' \otimes \phi'') = [D'(u)\phi'] \otimes [D''(u)\phi'']\,. \tag{183}$$

Denote this representation as $D(u) = D'(u) \otimes D''(u)$ and call it the **tensor product** of D' and D''. The matrix $D(u)$ is the Kronecker product of $D'(u)$ and $D''(u)$. For the characters, we clearly have

$$\chi(u) = \chi'(u)\chi''(u). \tag{184}$$

We can extend this tensor product definition to the product of any finite number of representations.

The tensor product of two irreps, $D^{j'}(u)$ and $D^{j''}(u)$, is in general not irreducible. We know however, that it is completely reducible, hence a direct sum of irreps:

$$D^{j'}(u) \otimes D^{j''}(u) = \sum_j \oplus C_{j'j''j} D^j(u). \tag{185}$$

This is called a "Clebsch-Gordan series". The $C_{j'j''j}$ coefficients are sometimes referred to as Clebsch-Gordan coefficients, although we tend to use that name for a different set of coefficients. These coefficients must, of course, be non-negative integers. We have the corresponding identity:

$$\chi_{j'}(u)\chi_{j''}(u) = \sum_j C_{j'j''j}\chi_j(u). \tag{186}$$

We now come to the important theorems on combining angular momenta in quantum mechanics:

Theorem:

1. There exists only a countably infinite number of inequivalent irreps of $SU(2)$. For every positive integer $(2j+1)$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, there exists precisely one irrep $D^j(u)$ (up to similarity transformations) of dimension $(2j+1)$. As $2j$ runs through all non-negative integers, the representations $D^j(u)$ exhaust the set of all equivalence classes.

2. The character of irrep $D^j(u)$ is:

$$\chi_j\left[u_{\mathbf{e}}(\theta)\right] = \frac{\sin\frac{2j+1}{2}\theta}{\sin\frac{1}{2}\theta}, \tag{187}$$

and

$$\chi_j(I) = d_j = 2j+1. \tag{188}$$

3. The representation $D^j(u)$ occurs precisely once in the reduction of the $2j$-fold tensor product $(\otimes u)^{2j}$, and we have:

$$\int_{SU(2)} d(u)\chi_j(u)\left[\mathrm{Tr}(u)\right]^{2j} = 1. \tag{189}$$

Proof: We take from $\int_{SU(2)} d(u)\chi_j^*(u)\chi_{j'}(u) = \delta_{jj'}$ the suggestion that the characters of the irreps are a complete set of orthonormal functions on a Hilbert space of class-functions of $SU(2)$ (A **class-function** is a function which takes the same value for every element in a conjugate class).

Consider the following function on $SU(2)$:

$$\omega(u) \equiv 1 - \frac{1}{8}\mathrm{Tr}\left[(u-I)^\dagger(u-I)\right] = \frac{1}{2}\left[1+\frac{1}{2}\mathrm{Tr}(u)\right], \tag{190}$$

which satisfies the conditions $1 > \omega(u) \geq 0$ for $u \neq I$, and $u(I) = 1$ (*e.g.*, noting that $u_{\mathbf{e}}(\theta) = \cos\frac{\theta}{2}I +$ a traceless piece). Hence, we have the lemma: If $f(u)$ is a continuous function on $SU(2)$, then

$$\lim_{n\to\infty}\frac{\int_{SU(2)} d(u)f(u)\left[\omega(u)\right]^n}{\int_{SU(2)} d(u)\left[\omega(u)\right]^n} = f(I). \tag{191}$$

The intuition behind this lemma is that, as $n \to \infty$, $\left[\omega(u)\right]^n$ becomes increasingly peaked about $u = I$.

Thus, if $D^j(u)$ is any irrep of $SU(2)$, then there exists an integer n such that:

$$\int_{SU(2)} d(u)\chi_j(u^{-1})\left[\mathrm{Tr}(u)\right]^n \neq 0. \tag{192}$$

Therefore, the irrep $D^j(u)$ occurs in the reduction of the tensor product $(\otimes u)^n$.

Next, we apply the Gram-Schmidt process to the infinite sequence $\{[\mathrm{Tr}(u)]^n \,|n = 0, 1, 2, \ldots\}$ of linearly independent class functions on $SU(2)$ to obtain the orthonormal sequence $\{B_n(u)|n = 0, 1, 2, \ldots\}$ of class functions:

$$\int_{SU(2)} d(u) B_n^*(u) B_m(u) = \frac{1}{\pi}\int_0^{2\pi} d\theta \sin^2\frac{\theta}{2} B_n^*\left[u_{e_3}(\theta)\right] B_m\left[u_{e_3}(\theta)\right] = \delta_{nm}, \tag{193}$$

where we have used the measure $d(u) = \frac{1}{4\pi^2} d\Omega_{\mathbf{e}} \sin^2\frac{\theta}{2} d\theta$ and the fact that, since B_n is a class function, it has the same value for a rotation by angle θ about any axis.

Now, write

$$\beta_n(\theta) = B_n\left[u_{e_3}(\theta)\right] = B_n\left[u_{\mathbf{e}}(\theta)\right]. \tag{194}$$

Noting that

$$[\mathrm{Tr}(u)]^n = \left(2\cos\frac{\theta}{2}\right)^n = \sum_{m=0}^{n} \binom{n}{m} e^{i\theta\left(\frac{n}{2}-m\right)}, \tag{195}$$

we may obtain the result

$$\beta_n(\theta) = \frac{\sin\left[(n+1)\theta/2\right]}{\sin\theta/2} = B_n\left[u_{\mathbf{e}}(\theta)\right] = B_n^*\left[u_{\mathbf{e}}(\theta)\right], \tag{196}$$

by adopting suitable phase conventions for the B's. Furthermore,

$$\int_{SU(2)} d(u) B_n^*(u) \left[\mathrm{Tr}(u)\right]^m = \begin{cases} 0 & \text{if } n > m, \\ 1 & \text{if } n = m. \end{cases} \tag{197}$$

We need to prove now that the functions $B_n(u)$ are characters of the irreps. We shall prove this by induction on n. First, $B_0(u) = 1$; $B_0(u)$ is the character of the trivial one-dimensional **identity representation**: $D(u) = 1$. Assume now that for some integer $n_0 \geq 0$ the functions $B_n(u)$ for $n = 0, 1, \ldots, n_0$ are all characters of irreps. Consider the reduction of the representation $(\otimes u)^{n_0+1}$:

$$[\mathrm{Tr}(u)]^{n_0+1} = \sum_{n=0}^{n_0} N_{n_0,n} B_n(u) + \sum_{j\in J_{n_0}} c_{n_0,j}\chi_j(u), \tag{198}$$

where $N_{n_0,n}$ and $c_{n_0,j}$ are integers ≥ 0, and the $c_{n_0,j}$ sum is over irreps D^j such that the characters are not in the set $\{B_n(u)|n = 0, 1, \ldots, n_0\}$ (j runs over a finite subset of the J_{n_0} index set).

With the above fact that:

$$\int_{SU(2)} d(u) B_n^*(u) \left[\mathrm{Tr}(u)\right]^{n_0+1} = \begin{cases} 0 & \text{if } n > n_0+1, \\ 1 & \text{if } n = n_0+1, \end{cases} \tag{199}$$

we have,

$$B_{n_0+1}(u) = \sum_{j\in J_{n_0}} c_{n_0,j}\chi_j(u). \tag{200}$$

Squaring both sides, and averaging over $SU(2)$ yields

$$1 = \sum_{j\in J_{n_0}} (c_{n_0,j})^2. \tag{201}$$

But the $c_{n_0,j}$ are integers, so there is only one term, with $c_{n_0,j} = 1$. Thus, $B_{n_0+1}(u)$ is a character of an irrep, and we see that there are a countably infinite number of (inequivalent) irreducible representations.

Let us obtain the dimensionality of the irreducible representation. The $B_n(u), n = 0, 1, 2, \ldots$ correspond to characters of irreps. We'll label these irredusible representations $D^j(u)$ according to the $1:1$ mapping $2j = n$. Then

$$\begin{aligned} d_j &= \chi_j(I) = B_{2j}(I) = \beta_{2j}(0) && (202)\\ &= \lim_{\theta\to 0} \frac{\sin\left[(2j+1)\theta/2\right]}{\sin\theta/2} && (203)\\ &= 2j+1. && (204) \end{aligned}$$

Next, we consider the reduction of the tensor product of two irreducible representations of $SU(2)$. This gives our "rule for combining angular momentum". That is, it tells us what angular momenta may be present in a system composed of two components with angular momenta. For example, it may be applied to the combination of a spin and an orbital angular momentum.

Theorem: Let j_1 and j_2 index two irreps of $SU(2)$. Then the Clebsch-Gordan series for the tensor product of these representations is:

$$D^{j_1}(u) \otimes D^{j_2}(u) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \oplus D^j(u). \tag{205}$$

Equivalently,

$$\chi_{j_1}(u)\chi_{j_2}(u) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \chi_j(u) = \sum_j C_{j_1j_2j}\chi_j(u), \tag{206}$$

where

$$\begin{aligned} C_{j_1j_2j} &= \int_{SU(2)} d(u)\chi_{j_1}(u)\chi_{j_2}(u)\chi_j(u) &(207)\\ &= \begin{cases} 1 & \text{iff } j_1+j_2+j \text{ is an integer, and a triangle can} \\ & \text{be formed with sides } j_1, j_2, j, \\ 0 & \text{otherwise.} \end{cases} &(208)\end{aligned}$$

The proof of this theorem is straightforward, by considering

$$C_{j_1j_2j} = \frac{1}{\pi}\int_0^{2\pi} d\theta \sin\left[(j_1+\frac{1}{2})\theta\right] \sin\left[(j_2+\frac{1}{2})\theta\right] \sum_{m=-j}^{j} e^{-im\theta}, \tag{209}$$

etc., as the reader is encouraged to carry out.

We have found all the irreps of $SU(2)$. Which are also irreps of $O^+(3)$? This is the subject of the next theorem:

Theorem: If $2j$ is an odd integer, then $D^j(u)$ is a faithful representation of $SU(2)$, and hence is not a representation of $O^+(3)$. If $2j > 0$ is an even integer (and hence $j > 0$ is an integer), then $R_{\mathbf{e}}(\theta) \to D^j[u_{\mathbf{e}}(\theta)]$ is a faithful representation of $O^+(3)$. Except for the trivial identity representation, all irreps of $O^+(3)$ are of this form.

The proof of this theorem is left to the reader.

We will not concern ourselves especially much with issues of constructing a proper Hilbert space, such as completeness, here. Instead, we'll concentrate on making the connection between $SU(2)$ representation theory and angular momentum in quantum mechanics a bit more concrete. We thus introduce the quantum mechanical angular momentum operators.

Theorem: Let $u \to U(u)$ be a (strongly-)continuous unitary representation of $SU(2)$ on Hilbert space $\mathcal{H}$. Then there exists a set of 3 self-adjoint operators $J_k, k = 1,2,3$ such that

$$U[u_{\mathbf{e}}(\theta)] = \exp(-i\theta\mathbf{e}\cdot\mathbf{J}). \tag{210}$$

To keep things simple, we'll consider now $U(u) = D(u)$, where $D(u)$ is a finite-dimensional representation – the appropriate extension to the general case may be demonstrated, but takes some care, and we'll omit it here.

1. The function $D[u_{\mathbf{e}}(\theta)]$ is (for $\mathbf{e}$ fixed), an infinitely differentiable function of θ. Define the matrices $J(\mathbf{e})$ by:

$$J(\mathbf{e}) \equiv i \left\{ \frac{\partial}{\partial\theta} D\left[u_{\mathbf{e}}(\theta)\right] \right\}_{\theta=0}. \tag{211}$$

Also, let $J_1 = J(\mathbf{e}_1)$, $J_2 = J(\mathbf{e}_2)$, $J_3 = J(\mathbf{e}_3)$. Then

$$J(\mathbf{e}) = \mathbf{e} \cdot \mathbf{J} = \sum_{k=1}^{3} (\mathbf{e} \cdot \mathbf{e}_k) J_k. \tag{212}$$

For any unit vector $\mathbf{e}$ and any θ, we have

$$D\left[u_{\mathbf{e}}(\theta)\right] = \exp(-i\theta \mathbf{e} \cdot \mathbf{J}). \tag{213}$$

The matrices J_k satisfy:

$$[J_k, J_\ell] = i\epsilon_{k\ell m} J_m. \tag{214}$$

The matrices $-iJ_k, k = 1, 2, 3$ form a basis for a representation of the Lie algebra of $O^+(3)$ under the correspondence:

$$\mathcal{J}_k \to -iJ_k, \quad k = 1, 2, 3. \tag{215}$$

The matrices J_k are hermitian if and only if $D(u)$ is unitary.

2. The matrices $J_k, k = 1, 2, 3$ form an irreducible set if and only if the representation $D(u)$ is irreducible. If $D(u) = D^j(u)$ is irreducible, then for any $\mathbf{e}$, the eigenvalues of $\mathbf{e} \cdot \mathbf{J}$ are the numbers $m = -j, -j+1, \ldots, j-1, j$, and each eigenvalue has multiplicity one. Furthermore:

$$\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2 = j(j+1). \tag{216}$$

3. For any representation $D(u)$ we have

$$D(u)\mathbf{J}^2 D(u^{-1}) = \mathbf{J}^2; \quad [J_k, \mathbf{J}^2] = 0, \quad k = 1, 2, 3. \tag{217}$$

Proof: (Partial) Consider first the case where $D(u) = D^j(u)$ is an irrep. Let $M_j = \{m|m = -j, \ldots, j\}$, let $\mathbf{e}$ be a fixed unit vector, and define the operators $F_m(\mathbf{e})$ (with $m \in M_j$) by:

$$F_m(\mathbf{e}) \equiv \frac{1}{4\pi}\int_0^{4\pi} d\theta e^{im\theta} D^j[u_{\mathbf{e}}(\theta)]. \tag{218}$$

Multiply this defining equation by $D^j[u_{\mathbf{e}}(\theta')]$:

$$\begin{aligned} D^j[u_{\mathbf{e}}(\theta')]F_m(\mathbf{e}) &= \frac{1}{4\pi}\int_0^{4\pi} d\theta e^{im\theta} D^j[u_{\mathbf{e}}(\theta+\theta')] & (219)\\ &= \frac{1}{4\pi}\int_0^{4\pi} d\theta e^{im(\theta-\theta')} D^j[u_{\mathbf{e}}(\theta)] & (220)\\ &= e^{(-im\theta')}F_m(\mathbf{e}). & (221) \end{aligned}$$

Thus, either $F_m(\mathbf{e}) = 0$, or $e^{(-im\theta')}$ is an eigenvalue of $D^j[u_{\mathbf{e}}(\theta')]$. But

$$\chi_j[u_{\mathbf{e}}(\theta)] = \sum_{n=-j}^{j} e^{-in\theta}, \tag{222}$$

and hence,

$$\mathrm{Tr}[F_m(\mathbf{e})] = \begin{cases} 1 & \text{if } m \in M_j, \\ 0 & \text{otherwise.} \end{cases} \tag{223}$$

Therefore, $F_m(\mathbf{e}) \neq 0$.

We see that the $\{F_m(\mathbf{e})\}$ form a set of $2j+1$ independent one-dimensional projection operators, and we can represent the $D^j(u)$ by:

$$D^j[u_{\mathbf{e}}(\theta)] = \sum_{m=-j}^{j} e^{-im\theta}F_m(\mathbf{e}). \tag{224}$$

From this, we obtain:

$$J(\mathbf{e}) \equiv i\left\{\frac{\partial}{\partial\theta}D^j\left[u_{\mathbf{e}}(\theta)\right]\right\}_{\theta=0} = \sum_{m=-j}^{j} mF_m(\mathbf{e}), \tag{225}$$

and

$$D^j[u_{\mathbf{e}}(\theta)] = \exp\left[-i\theta J(\mathbf{e})\right], \tag{226}$$

which is an entire function of θ for fixed $\mathbf{e}$.

Since every finite-dimensional continuous representation $D(u)$ is a direct sum of a finite number of irreps, this result holds for any such representation:

$$D[u_{\mathbf{e}}(\theta)] = \exp\left[-i\theta J(\mathbf{e})\right]. \tag{227}$$

Let $\mathbf{w}$ be a unit vector with components $w_k, k = 1,2,3$. Consider "small" rotations about $\mathbf{w}$:

$$u_{\mathbf{w}}(\theta) = u_{\mathbf{e}_1}(\theta w_1)u_{\mathbf{e}_2}(\theta w_2)u_{\mathbf{e}_3}(\theta w_3)u_{\mathbf{e}(\theta,\mathbf{w})}[\alpha(\theta,\mathbf{w})], \tag{228}$$

where $\alpha(\theta,\mathbf{w}) = O(\theta^2)$ for small θ. Thus,

$$\exp\left[-i\theta J(\mathbf{w})\right] = e^{-i\theta w_1 J_1}e^{-i\theta w_2 J_2}e^{-i\theta w_3 J_3}e^{-i\alpha J(\mathbf{e})}. \tag{229}$$

Expanding the exponentials and equating coefficients of terms linear in θ yields the result:

$$J(\mathbf{w}) = w_1 J_1 + w_2 J_2 + w_3 J_3 = \mathbf{w}\cdot\mathbf{J}. \tag{230}$$

To obtain the commutation relations, consider

$$u_{\mathbf{e}_1}(\theta)u_{\mathbf{e}_2}(\theta')u^{-1}_{\mathbf{e}_1}(\theta) \tag{231}$$

$$= \left(\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma_1\right)\left(\cos\frac{\theta'}{2}I - i\sin\frac{\theta'}{2}\sigma_2\right) \tag{232}$$

$$\left(\cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}\sigma_1\right) \tag{233}$$

$$= \cos\frac{\theta'}{2}I - i\sin\frac{\theta'}{2}\left(\cos\theta\sigma_2 + \sin\theta\sigma_3\right) \tag{234}$$

$$= u_{\mathbf{e}_2\cos\theta + \mathbf{e}_3\sin\theta}(\theta'). \tag{235}$$

Thus,

$$\exp(-i\theta J_1)\exp(-i\theta' J_2)\exp(i\theta J_1) = \exp\left[-i\theta'(J_2\cos\theta + J_3\sin\theta)\right]. \tag{236}$$

Expanding the exponentials, we have:

$$\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\theta^{n+m}\theta'^{\ell}\frac{J_1^n J_2^\ell J_1^m}{n!\ell!m!}(-i)^{n+\ell}i^m = \sum_{r=0}^{\infty}(-i)^r\theta'^r(J_2\cos\theta + J_3\sin\theta)^r\frac{1}{r!}. \tag{237}$$

We equate coefficients of the same powers of θ, θ'. In particular, the terms of order $\theta\theta'$ yield the result:

$$[J_1, J_2] = iJ_3. \tag{238}$$

We thus also have, for example:

$$\begin{aligned}
[J_1, \mathbf{J}^2] &= [J_1, J_1^2 + J_2^2 + J_3^2] && (239)\\
&= J_1J_2J_2 - J_2J_2J_1 + J_1J_3J_3 - J_3J_3J_1 && (240)\\
&= (iJ_3 + J_2J_1)J_2 - J_2(-iJ_3 + J_1J_2) + && (241)\\
&\quad (-iJ_2 + J_3J_1)J_3 - J_3(iJ_2 + J_1J_3) && (242)\\
&= 0. && (243)
\end{aligned}$$

As a consequence, we also have:

$$D(u)\mathbf{J}^2 D(u^{-1}) = e^{-i\theta\mathbf{e}\cdot\mathbf{J}}\mathbf{J}^2 e^{i\theta\mathbf{e}\cdot\mathbf{J}} = \mathbf{J}^2. \tag{244}$$

In particular, this is true for an irrep:

$$D^j(u)\mathbf{J}^2 D^j(u^{-1}) = \mathbf{J}^2. \tag{245}$$

Therefore $\mathbf{J}^2$ is a multiple of the identity (often referred to as a "casimir operator").

Let us determine the multiple. Take the trace:

$$\begin{aligned}
\mathrm{Tr}(\mathbf{J}^2) &= 3\mathrm{Tr}(J_3^2) && (246)\\
&= 3\mathrm{Tr}\left\{\left\{\frac{\partial}{\partial\theta}D^j\left[u_{\mathbf{e}_3}(\theta)\right]\right\}^2_{\theta=0}\right\} && (247)\\
&= -3\mathrm{Tr}\Bigg\{\lim_{\substack{\Delta\to 0\\ \Delta'\to 0}}\frac{1}{\Delta\Delta'}\Bigg\{\left[D^j\left(u_{\mathbf{e}_3}(\theta+\Delta)\right) - D^j\left(u_{\mathbf{e}_3}(\theta)\right)\right] && (248)\\
&\quad \left[D^j\left(u_{\mathbf{e}_3}(\theta+\Delta')\right) - D^j\left(u_{\mathbf{e}_3}(\theta)\right)\right]\Bigg\}\Bigg\}_{\theta=0} && (249)\\
&= -3\mathrm{Tr}\Bigg\{\lim_{\substack{\Delta\to 0\\ \Delta'\to 0}}\frac{1}{\Delta\Delta'}\Bigg\{\Big[D^j\left(u_{\mathbf{e}_3}(\Delta+\Delta')\right) && (250)\\
&\quad -D^j\left(u_{\mathbf{e}_3}(\Delta')\right) - D^j\left(u_{\mathbf{e}_3}(\Delta)\right) + D^j\left(u_{\mathbf{e}_3}(0)\right)\Big]\Bigg\}\Bigg\} && (251)
\end{aligned}$$

$$= \quad -3\mathrm{Tr}\left\{\left[\frac{\partial^2}{\partial\theta^2}D^j\left[u_{\mathbf{e}_3}(\theta)\right]\right]_{\theta=0}\right\} \tag{252}$$

$$= \quad -3\left\{\frac{\partial^2}{\partial\theta^2}\chi^j\left[u_{\mathbf{e}_3}(\theta)\right]\right\}_{\theta=0}. \tag{253}$$

There are different ways to evaluate this. We could insert

$$\chi^j\left[u_{\mathbf{e}_3}(\theta)\right] = \frac{\sin(j+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \tag{254}$$

or we could use

$$\chi^j\left[u_{\mathbf{e}_3}(\theta)\right] = \sum_{m=-j}^{j} e^{im\theta}. \tag{255}$$

In either case, we find:

$$\mathrm{Tr}(\mathbf{J}^2) = j(j+1)(2j+1). \tag{256}$$

Since $d_j = 2j+1$, this gives the familiar result:

$$\mathbf{J}^2 = j(j+1)I. \tag{257}$$

Finally, we'll compute the eigenvalues of $\mathbf{e}\cdot\mathbf{J} = J(\mathbf{e}) = \sum_{m=-j}^{j} mF_m(\mathbf{e})$. We showed earlier that the $F_m(\mathbf{e})$ are one-dimensional projection operators, and it thus may readily be demonstrated that

$$J(\mathbf{e})F_m(\mathbf{e}) = mF_m(\mathbf{e}). \tag{258}$$

Hence, the desired eigenvalues are $m = \{-j, -j+1, \ldots, j-1, j\}$.

11 Standard Conventions

Let's briefly summarize where we are. We'll pick some "standard" conventions towards building explicit representations for rotations in quantum mechanics.

- The rotation group in quantum mechanics is postulated to be $SU(2)$, with elements

  $$u = u_{\mathbf{e}}(\theta) = e^{-\frac{i}{2}\theta\mathbf{e}\cdot\boldsymbol{\sigma}}, \tag{259}$$

  describing a rotation by angle θ about vector $\mathbf{e}$, in the clockwise sense as viewed along $\mathbf{e}$.

- $O^+(3)$ is a two to one homomorphism of $SU(2)$:

$$R_{mn}(u) = \frac{1}{2}\text{Tr}(u^\dagger \sigma_m u \sigma_n). \tag{260}$$

- To every representation of $SU(2)$ there corresponds a representation of the Lie algrebra of $SU(2)$ given by the real linear span of the three matrices $-iJ_k, k = 1, 2, 3$, where

$$[J_m, J_n] = i\epsilon_{mnp} J_p. \tag{261}$$

  The vector operator $\mathbf{J}$ is interpreted as angular momentum. Its square is invariant under rotations.

- The matrix group $SU(2)$ is a representation of the abstract group $SU(2)$, and this representation is denoted $D^{\frac{1}{2}}(u) = u$. For this representation, $J_k = \frac{1}{2}\sigma_k$.

- Every finite dimensional representation of $SU(2)$ is equivalent to a unitary representation, and every unitary irreducible representation of $SU(2)$ is finite dimensional. Therefore, the generating operators, J_k, can always be chosen to be hermitian.

- Let $2j$ be a non-negative integer. To every $2j$ there corresponds a unique irrep by unitary transformations on a $2j+1$-dimensional carrier space, which we denote

$$D^j = D^j(u). \tag{262}$$

  These representations are constructed according to conventions which we take to define the "standard representations":
  The matrices J_k are hermitian and constructed according to the following: Let $|j, m\rangle, m = -j, -j+1, \ldots, j-1, j$ be a complete orthonormal basis in the carrier space such that:

$$\begin{aligned}
\mathbf{J}^2|j,m\rangle &= j(j+1)|j,m\rangle & (263)\\
J_3|j,m\rangle &= m|j,m\rangle & (264)\\
J_+|j,m\rangle &= \sqrt{(j-m)(j+m+1)}|j,m+1\rangle & (265)\\
J_-|j,m\rangle &= \sqrt{(j+m)(j-m+1)}|j,m-1\rangle, & (266)
\end{aligned}$$

where

$$J_\pm \equiv J_1 \pm iJ_2. \tag{267}$$

According to convention, matrices J_1 and J_3 are real, and matrix J_2 is pure imaginary. The matrix

$$D^j\left[u_{\mathbf{e}}(\theta)\right] = \exp(-i\theta\mathbf{e}\cdot\mathbf{J}), \tag{268}$$

describes a rotation by θ about unit vector $\mathbf{e}$.

- If $j = \frac{1}{2}$-integer, then $D^j(u)$ are faithful representations of $SU(2)$. If j is an integer, then $D^j(u)$ are representations of $O^+(3)$ (and are faithful if $j > 0$). Also, if j is an integer, then the representation $D^j(u)$ is similar to a representation by real matrices:

$$R_{mn}(u) = \frac{3}{j(j+1)(2j+1)}\mathrm{Tr}\left[D^{j\dagger}(u)J_m D^j(u)J_n\right]. \tag{269}$$

- In the standard basis, the matrix elements of $D^j(u)$ are denoted:

$$D^j_{m_1m_2}(u) = \langle j, m_1|D^j(u)|j, m_2\rangle, \tag{270}$$

and thus,

$$D^j(u)|j,m\rangle = \sum_{m'=-j}^{j} D^j_{m'm}(u)|j,m'\rangle. \tag{271}$$

Let $|\phi\rangle$ be an element in the carrier space of D^j, and let $|\phi'\rangle = D^j(u)|\phi\rangle$ be the result of applying rotation $D^j(u)$ to $|\phi\rangle$. We may expand these vectors in the basis:

$$|\phi\rangle = \sum_m \phi_m|j,m\rangle \tag{272}$$

$$|\phi'\rangle = \sum_m \phi'_m|j,m\rangle. \tag{273}$$

Then

$$\phi'_m = \sum_{m'} D^j_{mm'}(u)\phi_{m'}. \tag{274}$$

- Since matrices u and $D^j(u)$ are unitary,

$$D^j_{m_1m_2}(u^{-1}) = D^j_{m_1m_2}(u^\dagger) = D^{*j}_{m_2m_1}(u). \tag{275}$$

- The representation D^j is the symmetrized $(2j)$-fold tensor product of the representation $D^{\frac{1}{2}} = SU(2)$ with itself. For the standard representation, this is expressed by an explicit formula for matrix elements $D^j_{m_1 m_2}(u)$ as polynomials of degree $2j$ in the matrix elements of $SU(2)$: Define the quantities $D^j_{m_1 m_2}(u)$ for $m_1, m_2 = -j, \ldots, j$ by:

$$\begin{aligned} \langle \lambda^* | u | \eta \rangle^{2j} &= (\lambda_1 u_{11} \eta_1 + \lambda_1 u_{12} \eta_2 + \lambda_2 u_{21} \eta_1 + \lambda_2 u_{22} \eta_2)^{2j} && (276) \\ &= (2j)! \sum_{m_1, m_2} && (277) \\ &\quad \frac{\lambda_1^{j+m_1} \lambda_2^{j-m_1} \eta_1^{j+m_2} \eta_2^{j-m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} D^j_{m_1 m_2}(u). \end{aligned}$$

  We defer to later the demonstration that the matrix elements $D^j_{m_1 m_2}(u)$ so defined are identical with the earlier definition for the standard representation. A consequence of this formula (which the reader is encouraged to demonstrate) is that, in the standard representation,

$$\begin{aligned} D^j(u^*) &= D^{*j}(u) && (278) \\ D^j(u^T) &= D^{jT}(u). && (279) \end{aligned}$$

  Also, noting that $u^* = \sigma_2 u \sigma_2$, we obtain

$$D^{*j}(u) = \exp(-i\pi J_2) D^j(u) \exp(i\pi J_2), \quad (280)$$

  making explicit our earlier statement that the congugate representation was equivalent in $SU(2)$ [We remark that this property does not hold for $SU(n)$, if $n > 2$].

- We can also describe the standard representation in terms of an action of the rotation group on homogeneous polynomials of degree $2j$ of complex variables x and y. We define, for each $j = 0, \frac{1}{2}, 1, \ldots$, and $m = -j, \ldots, j$ the polynomial:

$$P_{jm}(x, y) \equiv \frac{x^{j+m} y^{j-m}}{\sqrt{(j+m)!(j-m)!}}; \quad P_{00} \equiv 1. \quad (281)$$

  We also define $P_{jm} \equiv 0$ if $m \notin \{-j, \ldots, j\}$. In addition, define the differential operators $J_k, k = 1, 2, 3, J_+, J_-$:

$$J_3 = \frac{1}{2}(x\partial_x - y\partial_y) \quad (282)$$

$$
\begin{aligned}
J_1 &= \frac{1}{2}(x\partial_y + y\partial_x) = \frac{1}{2}(J_+ + J_-) && (283)\\
J_2 &= \frac{i}{2}(y\partial_x - x\partial_y) = \frac{i}{2}(J_- - J_+) && (284)\\
J_+ &= x\partial_y = J_1 + iJ_2 && (285)\\
J_- &= y\partial_x = J_1 - iJ_2 && (286)
\end{aligned}
$$

These definitions give

$$
\mathbf{J}^2 = \frac{1}{4}\left[(x\partial_x - y\partial_y)^2 + 2(x\partial_x + y\partial_y)\right]. \tag{287}
$$

We let these operators act on our polynomials:

$$
\begin{aligned}
J_3 P_{jm}(x,y) &= \left[\frac{1}{2}(x\partial_x - y\partial_y)\right]\frac{x^{j+m}y^{j-m}}{\sqrt{(j+m)!(j-m)!}} && (288)\\
&= \frac{1}{2}[j+m-(j-m)]P_{jm}(x,y) && (289)\\
&= mP_{jm}(x,y). && (290)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_+ P_{jm}(x,y) &= (x\partial_y)\frac{x^{j+m}y^{j-m}}{\sqrt{(j+m)!(j-m)!}} && (291)\\
&= (j-m)\frac{x^{j+m+1}y^{j-m-1}}{\sqrt{(j+m)!(j-m)!}} && (292)\\
&= (j-m)\sqrt{\frac{(j+m+1)!(j-m-1)!}{(j+m)!(j-m)!}}P_{j,m+1}(x,y) \\
&= \sqrt{(j-m)(j+m+1)}P_{j,m+1}(x,y). && (293)
\end{aligned}
$$

Likewise,

$$
J_- P_{jm}(x,y) = \sqrt{(j+m)(j-m+1)}P_{j,m-1}(x,y), \tag{294}
$$

and

$$
\mathbf{J}^2 P_{jm}(x,y) = j(j+1)P_{jm}(x,y). \tag{295}
$$

We see that the actions of these differential operators on the monomials, P_{jm}, are according to the standard representation of the Lie algebra of the rotation group (that is, we compare with the actions of the standard representation for $\mathbf{J}$ on orthonormal basis $|j,m\rangle$).

Thus, regarding $P_{jm}(x,y)$ as our basis, a rotation corresponds to:

$$D^j(u)P_{jm}(x,y) = \sum_{m'} D^j_{m'm}(u)P_{jm'}(x,y). \tag{296}$$

Now,

$$\begin{aligned} D^{\frac{1}{2}}(u)P_{\frac{1}{2}m}(x,y) &= \sum_{m'} D^{\frac{1}{2}}_{m'm}(u)P_{\frac{1}{2}m'}(x,y) &(297)\\ &= \sum_{m'} u_{m'm}(u)P_{\frac{1}{2}m'}(x,y). &(298)\end{aligned}$$

Or,

$$uP_{\frac{1}{2}m}(x,y) = u_{\frac{1}{2}m}P_{\frac{1}{2}\frac{1}{2}}(x,y) + u_{-\frac{1}{2}m}P_{\frac{1}{2}-\frac{1}{2}}(x,y). \tag{299}$$

With $P_{\frac{1}{2}\frac{1}{2}}(x,y) = x$, and $P_{\frac{1}{2}-\frac{1}{2}}(x,y) = y$, we thus have (using normal matrix indices now on u)

$$\begin{aligned} uP_{\frac{1}{2}\frac{1}{2}}(x,y) &= u_{11}x + u_{21}y, &(300)\\ uP_{\frac{1}{2}-\frac{1}{2}}(x,y) &= u_{12}x + u_{22}y. &(301)\end{aligned}$$

Hence,

$$\begin{aligned} D^j(u)P_{jm}(x,y) &= P_{jm}(u_{11}x + u_{21}y, u_{12}x + u_{22}y) &(302)\\ &= \sum_{m'} D^j_{m'm}(u)P_{jm'}(x,y). &(303)\end{aligned}$$

Any homogeneous polynomial of degree $2j$ in (x,y) can be written as a unique linear combination of the monomials $P_{jm}(x,y)$. Therefore, the set of all such polynomials forms a vector space of dimension $2j+1$, and carries the standard representation D^j of the rotation group if the action of the group elements on the basis vectors P_{jm} is as above. Note that

$$\begin{aligned} P_{jm}(\partial_x,\partial_y)P_{jm'}(x,y) &= \frac{\partial_x^{j+m}\partial_y^{j-m}x^{j+m'}y^{j-m'}}{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}} \\ &= \delta_{mm'}. &(304)\end{aligned}$$

Apply this to

$$P_{jm}(u_{11}x + u_{21}y, u_{12}x + u_{22}y) = \sum_{m'} D^{\frac{1}{2}}_{m'm}(u) P_{\frac{1}{2}m'}(x, y) : \tag{305}$$

$$P_{jm}(\partial_x, \partial_y) \sum_{m''} D^{\frac{1}{2}}_{m''m'}(u) P_{\frac{1}{2}m''}(x, y) = D^j_{mm'}(u). \tag{306}$$

Hence,

$$D^j_{mm'}(u) = P_{jm}(\partial_x, \partial_y) P_{jm'}(u_{11}x + u_{21}y, u_{12}x + u_{22}y), \tag{307}$$

and we see that $D^j_{mm'}(u)$ is a homogeneous polynomial of degree $2j$ in the matrix elements of u.

Now,

$$\sum_{m=-j}^{j} P_{jm}(x_1, y_1) P_{jm}(x_2, y_2) = \sum_{m=-j}^{j} \frac{(x_1 x_2)^{j+m} (y_1 + y_2)^{j-m}}{(j+m)!(j-m)!}. \tag{308}$$

Using the binomial theorem, we can write:

$$\frac{(x_1 x_2 + y_1 y_2)^{2j}}{(2j)!} = \sum_{m=-j}^{j} \frac{(x_1 x_2)^{j+m} (y_1 + y_2)^{j-m}}{(j+m)!(j-m)!}. \tag{309}$$

Thus,

$$\sum_{m=-j}^{j} P_{jm}(x_1, y_1) P_{jm}(x_2, y_2) = \frac{(x_1 x_2 + y_1 y_2)^{2j}}{(2j)!}. \tag{310}$$

One final step remains to get our asserted equation defining the $D^j(u)$ standard representation in terms of u:

$$\sum_{m_1, m_2} \frac{\lambda_1^{j+m_1} \lambda_2^{j-m_1} \eta_1^{j+m_2} \eta_2^{j-m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} D^j_{m_1 m_2}(u) \tag{311}$$

$$= \sum_{m_1, m_2} P_{jm_1}(\lambda_1, \lambda_2) D^j_{m_1 m_2}(u) P_{jm_2}(\eta_1, \eta_2) \tag{312}$$

$$= \sum_{m_2} P_{jm_2}(u_{11}\lambda_1 + u_{21}\lambda_2, u_{12}\lambda_1 + u_{22}\lambda_2) P_{jm_2}(\eta_1, \eta_2) \tag{313}$$

$$= \frac{1}{(2j)!} (\lambda_1 u_{11} \eta_1 + \lambda_1 u_{12} \eta_2 + \lambda_2 u_{21} \eta_1 + \lambda_2 u_{22} \eta_2)^{2j}. \tag{314}$$

The step in obtaining Eqn. 313 follows from Eqn. 303, or it can be demonstrated by an explicit computation. Thus, we have now demonstrated our earlier formula, Eqn. 277, for the standard representation for D^j.

12 "Special" Cases

We have obtained a general expression for the rotation matrices for an irrep. Let us consider some "special cases", and derive some more directly useful formulas for the matrix elements of the rotation matrices.

1. Consider, in the standard representation, a rotation by angle π about the coordinate axes. Let $\rho_1 = \exp(-i\pi\sigma_1/2) = -i\sigma_1$. Using

$$D^j_{mm'}(u) = P_{jm}(\partial_x, \partial_y)P_{jm'}(u_{11}x + u_{21}y, u_{12}x + u_{22}y), \tag{315}$$

we find:

$$\begin{aligned} D^j_{mm'}(\rho_1) &= P_{jm}(\partial_x, \partial_y)P_{jm'}(-iy, -ix), & (316)\\ &= P_{jm}(\partial_x, \partial_y)(-)^j P_{jm'}(x, y), & (317)\\ &= e^{-i\pi j}\delta_{mm'}. & (318)\end{aligned}$$

Hence,

$$\exp(-i\pi J_1)|j, m\rangle = e^{-i\pi j}|j, -m\rangle. \tag{319}$$

Likewise, we define

$$\begin{aligned} \rho_2 &= \exp(-i\pi\sigma_2/2) = -i\sigma_2, & (320)\\ \rho_3 &= \exp(-i\pi\sigma_3/2) = -i\sigma_3, & (321)\end{aligned}$$

which have the properties:

$$\begin{aligned} \rho_1\rho_2 &= -\rho_2\rho_1 = \rho_3, & (322)\\ \rho_2\rho_3 &= -\rho_3\rho_2 = \rho_1, & (323)\\ \rho_3\rho_1 &= -\rho_1\rho_3 = \rho_2, & (324)\end{aligned}$$

and hence,

$$D^j(\rho_2) = D^j(\rho_3)D^j(\rho_1). \tag{325}$$

In the standard representation, we already know that

$$\exp(-i\pi J_3)|j, m\rangle = e^{-i\pi m}|j, m\rangle. \tag{326}$$

Therefore,

$$\begin{aligned} \exp(-i\pi J_2)|j, m\rangle &= \exp(-i\pi J_3)\exp(-i\pi J_1)|j, m\rangle & (327)\\ &= \exp(-i\pi J_3)\exp(-i\pi j)|j, -m\rangle & (328)\\ &= \exp(-i\pi(j - m))|j, -m\rangle. & (329)\end{aligned}$$

2. Consider the parameterization by Euler angles ψ, θ, ϕ:

$$u = e^{\psi \mathcal{J}_3} e^{\theta \mathcal{J}_2} e^{\phi \mathcal{J}_3}, \tag{330}$$

(here $\mathcal{J}_k = -\frac{i}{2}\sigma_k$) or,

$$D^j(u) = D^j(\psi, \theta, \phi) = e^{-i\psi J_3} e^{-i\theta J_2} e^{-i\phi J_3}, \tag{331}$$

where it is sufficient (for all elements of $SU(2)$) to choose the range of parameters:

$$0 \leq \psi < 2\pi, \tag{332}$$
$$0 \leq \theta \leq \pi, \tag{333}$$
$$0 \leq \phi < 4\pi \text{ (or } 2\pi\text{, if } j \text{ is integral).} \tag{334}$$

We define the functions

$$D^j_{m_1 m_2}(\psi, \theta, \phi) = e^{-i(m_1\psi + m_2\phi)} d^j_{m_1 m_2}(\theta) = \langle j, m_1 | D^j(u) | j, m_2 \rangle, \tag{335}$$

where we have introduced the real functions $d^j_{m_1 m_2}(\theta)$ given by:

$$d^j_{m_1 m_2}(\theta) \equiv D^j_{m_1 m_2}(0, \theta, 0) = \langle j, m_1 | e^{-i\theta J_2} | j, m_2 \rangle. \tag{336}$$

The "big-D" and "little-d" functions are useful in solving quantum mechanics problems involving angular momentum. The little-d functions may be found tabulated in various tables, although we have built enough tools to compute them ourselves, as we shall shortly demonstrate. Note that the little-d functions are real.

Here are some properties of the little-d functions, which the reader is encouraged to prove:

$$d^j_{m_1 m_2}(\theta) = d^{j*}_{m_1 m_2}(\theta) \tag{337}$$
$$= (-)^{m_1 - m_2} d^j_{m_2 m_1}(\theta) \tag{338}$$
$$= (-)^{m_1 - m_2} d^j_{-m_1, -m_2}(\theta) \tag{339}$$
$$d^j_{m_1 m_2}(\pi - \theta) = (-)^{j - m_2} d^j_{-m_1 m_2}(\theta) \tag{340}$$
$$= (-)^{j + m_1} d^j_{m_1, -m_2}(\theta) \tag{341}$$
$$d^j_{m_1 m_2}(-\theta) = d^j_{m_2 m_1}(\theta) \tag{342}$$
$$d^j_{m_1 m_2}(2\pi + \theta) = (-)^{2j} d^j_{m_1 m_2}(\theta). \tag{343}$$

The d^j functions are homogeneous polynomials of degree $2j$ in $\cos(\theta/2)$ and $\sin(\theta/2)$. Note that slightly different conventions from those here are sometimes used for the big-D and little-d functions.

The $D^j_{m_1m_2}(u)$ functions form a complete and essentially orthonormal basis of the space of square integrable functions on $SU(2)$:

$$\int_{SU(2)} d(u) D^{*j}_{m_1m_2}(u) D^{j'}_{m_1'm_2'}(u) = \frac{\delta_{jj'}\delta_{m_1m_1'}\delta_{m_2m_2'}}{2j+1}. \tag{344}$$

In terms of the Euler angles, $d(u) = \frac{1}{16\pi^2} d\psi \sin\theta d\theta d\phi$, and

$$\frac{\delta_{jj'}}{2j+1} = \frac{1}{16\pi^2}\int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta d\theta \int_0^{4\pi} d\phi \tag{345}$$

$$e^{i(m_1\psi+m_2\phi)} e^{-i(m_1\psi+m_2\phi)} d^j_{m_1m_2}(\theta) d^{j'}_{m_1m_2}(\theta) \tag{346}$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta d\theta d^j_{m_1m_2}(\theta) d^{j'}_{m_1m_2}(\theta). \tag{347}$$

3. Spherical harmonics and Legendre polynomials: The $Y_{\ell m}$ functions are special cases of the D^j. Hence we may *define*:

$$Y_{\ell m}(\theta,\psi) \equiv \sqrt{\frac{2\ell+1}{4\pi}} D^{*\ell}_{m0}(\psi,\theta,0), \tag{348}$$

where ℓ is an integer, and $m \in \{-\ell, -\ell+1, \ldots, \ell\}$.

This is equivalent to the standard definition, where the $Y_{\ell m}$'s are constructed to transform under rotations like $|\ell m\rangle$ basis vectors, and where

$$Y_{\ell 0}(\theta,0) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta). \tag{349}$$

According to our definition,

$$Y_{\ell 0}(\theta,0) = \sqrt{\frac{2\ell+1}{4\pi}} D^{*\ell}_{00}(0,\theta,0) = \sqrt{\frac{2\ell+1}{4\pi}} d^\ell_{00}(\theta). \tag{350}$$

Thus, we need to show that $d^\ell_{00}(\theta) = P_\ell(\cos\theta)$. This may be done by comparing the generating function for the Legendre polynomials:

$$\sum_{\ell=0}^{\infty} t^\ell P_\ell(x) = 1/\sqrt{1-2tx+t^2}, \tag{351}$$

with the generating function for $d_{00}(\theta)$, obtained from considering the generating function for the characters of the irreps of $SU(2)$:

$$\sum_j \chi_j(u) z^{2j} = 1/(1-2\tau z+z^2), \tag{352}$$

where $\tau = \mathrm{Tr}(u/2)$. We haven't discussed this generating function, and we leave it to the reader to pursue this demonstration further.

The other aspect of our assertion of equivalence requiring proof concerns the behavior of our spherical harmonics under rotations. Writing $Y_{\ell m}(\mathbf{e}) = Y_{\ell m}(\theta,\psi)$, where θ,ψ are the polar angles of unit vector $\mathbf{e}$, we can express our definition in the form:

$$Y_{\ell m}[R(u)\mathbf{e}_3] = \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell}_{0m}(u^{-1}), \tag{353}$$

since,

$$\begin{aligned}
Y_{\ell m}[R(u)\mathbf{e}_3] &= Y_{\ell m}(\mathbf{e}) = Y_{\ell m}(\theta,\psi) && (354)\\
&= \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell}_{m0}(\psi,\theta,0) && (355)\\
&= \sqrt{\frac{2\ell+1}{4\pi}} \langle j,m|D^{\ell}(u)|j,0\rangle^* && (356)\\
&= \sqrt{\frac{2\ell+1}{4\pi}} \langle D^{\ell}(u)(j,0)||j,m\rangle && (357)\\
&= \sqrt{\frac{2\ell+1}{4\pi}} \langle j,0|D^{\dagger\ell}(u)|j,m\rangle. && (358)
\end{aligned}$$

To any $u_0 \in SU(2)$ corresponds $\hat{R}(u_o)$ on any function $f(\mathbf{e})$ on the unit sphere as follows:

$$\hat{R}(u_o) f(\mathbf{e}) = f\left[R^{-1}(u_0)\mathbf{e}\right]. \tag{359}$$

Thus,

$$\begin{aligned}
\hat{R}(u_o) Y_{\ell m}(\mathbf{e}) &= Y_{\ell m}\left[R^{-1}(u_0)\mathbf{e}\right] && (360)\\
&= Y_{\ell m}\left[R^{-1}(u_0)R(u)\mathbf{e}_3\right] && (361)
\end{aligned}$$

$$
\begin{aligned}
&= Y_{\ell m}\left[R(u_0^{-1}u)\mathbf{e}_3\right] && (362)\\
&= \sqrt{\frac{2\ell+1}{4\pi}}D^{\ell}_{0m}(u^{-1}u_0) && (363)\\
&= \sqrt{\frac{2\ell+1}{4\pi}}\sum_{m'} D^{\ell}_{0m'}(u^{-1})D^{\ell}_{m'm}(u_0) && (364)\\
&= \sum_{m'} D^{\ell}_{m'm}(u_0)Y_{\ell m'}(\mathbf{e}). && (365)
\end{aligned}
$$

This shows that the $Y_{\ell m}$ transform under rotations according to the $|\ell, m\rangle$ basis vectors.

We immediately have the following properties:

(a) If $\mathbf{J} = -i\mathbf{x} \times \nabla$ is the angular momentum operator, then

$$
\begin{aligned}
J_3 Y_{\ell m}(\mathbf{e}) &= mY_{\ell m}(\mathbf{e}) && (366)\\
\mathbf{J}^2 Y_{\ell m}(\mathbf{e}) &= \ell(\ell+1)Y_{\ell m}(\mathbf{e}). && (367)
\end{aligned}
$$

(b) From th D^*D orthogonality relation, we further have:

$$
\int_0^{2\pi} d\psi \int_0^{\pi} d\theta \sin\theta Y^*_{\ell m}(\theta,\psi)Y_{\ell' m'}(\theta,\psi) = \delta_{mm'}\delta_{\ell\ell'}. \qquad (368)
$$

The $Y_{\ell m}(\theta,\psi)$ form a complete orthonormal set in the space of square-integrable functions on the unit sphere.

We give a proof of completeness here: If $f(\theta,\psi)$ is square-integrable, and if

$$
\int_0^{2\pi} d\psi \int_0^{\pi} d\theta \sin\theta f(\theta,\psi)Y^*_{\ell m}(\theta,\psi) = 0,\ \forall(\ell,m), \qquad (369)
$$

we must show that this means that the integral of $|f|^2$ vanishes. This follows from the completeness of $D^j(u)$ on $SU(2)$: Extend the domain of definition of f to $F(u) = F(\psi,\theta,\phi) = f(\theta,\psi)$, $\forall u \in SU(2)$. Then,

$$
\int_{SU(2)} d(u)F(u)D^j_{mm'}(u) = \begin{cases} 0 & \text{if } m' = 0, \text{ by assumption} \\ 0 & \text{if } m' \neq 0 \text{ since } F(u) \text{ is independent} \\ & \text{of } \phi, \text{ and } \int_0^{4\pi} d\phi e^{-im'\phi} = 0. \end{cases} \qquad (370)
$$

Hence, $\int_{SU(2)} d(u)|F(u)|^2 = 0$ by the completeness of $D^j(u)$, and therefore $\int_{(4\pi)} d\Omega |f|^2 = 0$.

(c) We recall that $Y_{\ell 0}(\theta, 0) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$. With

$$R(u_o)Y_{\ell m}(\mathbf{e}) = Y_{\ell m}\left[R^{-1}(u_0)\mathbf{e}\right] = \sum_{m'} D^\ell_{m'm}(u_0)Y_{\ell m'}(\mathbf{e}), \tag{371}$$

we have, for $m = 0$,

$$R(u_o)Y_{\ell 0}(\mathbf{e}) = Y_{\ell 0}\left[R^{-1}(u_0)\mathbf{e}\right] = \sum_{m'} D^\ell_{m'0}(u_0)Y_{\ell m'}(\mathbf{e}) \tag{372}$$

$$= \sqrt{\frac{4\pi}{2\ell+1}} \sum_{m'} Y^*_{\ell m'}(\mathbf{e}')Y_{\ell m'}(\mathbf{e}), \tag{373}$$

where we have defined $R(u_0)\mathbf{e}_3 = \mathbf{e}'$. But

$$\mathbf{e}\cdot\mathbf{e}' = \mathbf{e}\cdot[R(u_0)\mathbf{e}_3 = [R^{-1}(u_0)\mathbf{e}]\cdot\mathbf{e}_3, \tag{374}$$

and

$$Y_{\ell 0}[R^{-1}(u_0)\mathbf{e}] = \text{function of } \theta[R^{-1}(u_0)\mathbf{e}] \text{ only}, \tag{375}$$

$$= \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta), \tag{376}$$

where $\cos\theta = \mathbf{e}\cdot\mathbf{e}'$. Thus, we have the "addition theorem" for spherical harmonics:

$$P_\ell(\mathbf{e}\cdot\mathbf{e}') = \frac{4\pi}{2\ell+1}\sum_m Y^*_{\ell m}(\mathbf{e}')Y_{\ell m}(\mathbf{e}). \tag{377}$$

(d) In momentum space, we can therefore write

$$\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \frac{\delta(p-q)}{4\pi pq}\sum_{\ell=0}^\infty (2\ell+1)P_\ell(\cos\theta) \tag{378}$$

$$= \frac{\delta(p-q)}{4\pi pq}\sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta_p,\psi_p)Y^*_{\ell m}(\theta_q,\psi_q), \tag{379}$$

where $p = |\mathbf{p}|$, and θ is the angle between $\mathbf{p}$ and $\mathbf{q}$.

(e) Let us see now how we may compute the $d^j(\theta)$ functions. Recall

$$D^j_{mm'}(u) = P_{jm}(\partial_x, \partial_y)P_{jm'}(u_{11}x + u_{21}y, u_{12}x + u_{22}y). \tag{380}$$

With

$$u(0,\theta,0) = e^{-i\frac{\theta}{2}\sigma_2} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma_2 \tag{381}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{382}$$

we have:

$$d^j_{mm'}(\theta) = D^j_{mm'}(0,\theta,0) \tag{383}$$

$$= P_{jm}(\partial_x,\partial_y)P_{jm'}(x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2}, -x\sin\frac{\theta}{2} + y\cos\frac{\theta}{2}) \tag{384}$$

$$= \frac{\partial_x^{j+m}\partial_y^{j-m}(x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2})^{j+m'}(-x\sin\frac{\theta}{2} + y\cos\frac{\theta}{2})^{j-m'}}{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}. \tag{385}$$

Thus, we have an explicit means to compute the little-d functions. An alternate equation, which is left to the reader to derive (again using the P_{jm} functions), is

$$d^j_{mm'}(\theta) = \frac{1}{2\pi}\sqrt{\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}} \tag{386}$$

$$\times \int_0^{2\pi} d\alpha e^{i(m-m')\alpha}(\cos\frac{\theta}{2} + e^{i\alpha}\sin\frac{\theta}{2})^{j+m'}(\cos\frac{\theta}{2} - e^{-i\alpha}\sin\frac{\theta}{2})^{j-m}$$

In tabulating the little-d functions it is standard to use the labeling:

$$d^j = \begin{pmatrix} d^j_{jj} & \dots & d^j_{j,-j} \\ \vdots & \ddots & \vdots \\ d^j_{-j,j} & \dots & d^j_{-j,-j} \end{pmatrix}. \tag{387}$$

For example, we find:

$$d^{\frac{1}{2}}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{388}$$

and

$$d^1(\theta) = \begin{pmatrix} \frac{1}{2}(1+\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta) \\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}(1-\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta) \end{pmatrix}. \tag{389}$$

13 Clebsch-Gordan (Vector Addition) Coefficients

We consider the tensor (or Kronecker) product of two irreps, and its reduction according to the Clebsch-Gordan series:

$$D^{j_1}(u) \otimes D^{j_2}(u) = \sum_{j=|j_1-j_2|}^{j_1+j_2} D^j(u). \tag{390}$$

The carrier space of the representation $D^{j_1} \otimes D^{j_2}$ is of dimension $(2j_1+1)(2j_2+1)$. It is the tensor product of the carrier spaces for the representations D^{j_1} and D^{j_2}. Corresponding to the above reduction of the kronecker product, we have a decomposition of the carrier space into orthogonal subspaces carrying the representations D^j.

For each j we can select a standard basis system:

$$|j_1 j_2; j, m\rangle = \sum_{m_1, m_2} C(j_1 j_2 j; m_1 m_2 m)|j_1, m_1; j_2, m_2\rangle, \tag{391}$$

where the latter ket is just the tensor product of the standard basis vectors $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$. The coefficients C are the "vector-addition coefficients", or "Wigner coefficients", or, more commonly now, "Clebsch-Gordan (CG) coefficients". These coefficients must be selected so that the unit vectors $|j_1 j_2; j, m\rangle$ transform under rotations according to the standard representation.

We notice that the CG coefficients relate two systems of orthonormal basis vectors. Hence, they are matrix elements of a unitary matrix with rows labeled by the (m_1, m_2) pair, and columns labeled by the (j, m) pair. Thus, we have the orthonormality relations:

$$\sum_{m_1, m_2} C^*(j_1 j_2 j; m_1 m_2 m) C(j_1 j_2 j'; m_1 m_2 m') = \delta_{jj'} \delta_{mm'}, \tag{392}$$

$$\sum_{j,m} C^*(j_1 j_2 j; m_1 m_2 m) C(j_1 j_2 j; m_1' m_2' m) = \delta_{m_1 m_1'} \delta_{m_2 m_2'}. \tag{393}$$

The inverse basis transformation is:

$$|j_1, m_1; j_2, m_2\rangle = \sum_{j,m} C^*(j_1 j_2 j; m_1 m_2 m)|j_1 j_2; j, m\rangle. \tag{394}$$

We wish to learn more about the CG coefficients, including how to compute them in general. Towards accomplishing this, evaluate the matrix elements of the Clebsch-Gordan series, with

$$D^{j_1}_{m'_1 m''_1}(u) = \langle j_1, m'_1 | D^{j_1}(u) | j_1, m''_1 \rangle, \tag{395}$$

etc. Thus we obtain the explicit reduction formula for the matrices of the standard representations:

$$\begin{aligned} D^{j_1}_{m'_1 m''_1}(u) D^{j_2}_{m'_2 m''_2}(u) &= \langle j_1, m'_1; j_2, m'_2 | \sum_j D^j(u) | j_1, m''_1; j_2, m''_2 \rangle \qquad (396) \\ = \sum_j \sum_{m'} C(j_1 j_2 j; m'_1 m'_2 m') &\langle j_1 j_2; j, m' | D^j(u) \sum_{m''} C^*(j_1 j_2 j; m''_1 m''_2 m'') | j_1 j_2; j, m'' \rangle \\ = \sum_{j,m',m''} C(j_1 j_2 j; m'_1 m'_2 m') &C^*(j_1 j_2 j; m''_1 m''_2 m'') D^j_{m' m''}(u). \qquad (397) \end{aligned}$$

Next, we take this equation, multiply by $D^{*j}_{m'm''}(u)$, integrate over $SU(2)$, and use orthogonality to obtain:

$$C(j_1 j_2 j; m'_1 m'_2 m') C^*(j_1 j_2 j; m''_1 m''_2 m'') = (2j+1) \int_{SU(2)} d(u) D^{*j}_{m'm''}(u) D^{j_1}_{m'_1 m''_1}(u) D^{j_2}_{m'_2 m''_2}(u). \tag{398}$$

Thus, the CG coefficients for the standard representations are determined by the matrices in the standard representations, except for a phase factor which depends on $(j_1 j_2 j)$ [but not on $(m_1 m_2 m)$].

The coefficients vanish *unless*:

1. $j \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2\}$ (from the Clebsch-Gordan series).
2. $m = m_1 + m_2$ (as will be seen anon).
3. $m \in \{-j, -j+1, \ldots, j\}$, $m_1 \in \{-j_1, -j_1+1, \ldots, j_1\}$, $m_2 \in \{-j_2, -j_2 + 1, \ldots, j_2\}$ (by convention).

Consider the Euler angle parameterization for the matrix elements, giving:

$$\begin{aligned} &C(j_1 j_2 j; m'_1 m'_2 m') C^*(j_1 j_2 j; m''_1 m''_2 m'') \\ &\quad = (2j+1) \frac{1}{16\pi^2} \int_0^{2\pi} d\psi \int_0^{4\pi} d\phi \int_0^{\pi} \sin\theta d\theta \qquad (399) \\ &\quad \exp\{-i[(-m' + m'_1 + m'_2)\psi + (-m'' + m''_1 + m''_2)\phi]\} d^j_{m'm''}(\theta) d^{j_1}_{m'_1 m''_1}(\theta) d^{j_2}_{m'_2 m''_2}(\theta). \end{aligned}$$

We see that this is zero unless $m' = m'_1 + m'_2$ and $m'' = m''_1 + m''_2$, verifying the above assertion. Hence, we have:

$$\begin{aligned} C(j_1 j_2 j; m'_1 m'_2 m') C^*(j_1 j_2 j; m''_1 m''_2 m'') \\ = \frac{(2j+1)}{2} \int_0^\pi \sin\theta d\theta d^j_{m'm''}(\theta) d^{j_1}_{m'_1 m''_1}(\theta) d^{j_2}_{m'_2 m''_2}(\theta). \end{aligned} \tag{400}$$

Now, to put things in a more symmetric form, let $m'_3 = -m'$, $m''_3 = -m''$, and use

$$d^{j_3}_{-m'_3,-m''_3}(\theta) = (-)^{m'_3 - m''_3} d^{j_3}_{m'_3,m''_3}(\theta), \tag{401}$$

to obtain the result:

$$\begin{aligned} \frac{2}{2j+1}(-)^{m'_3 - m''_3} C(j_1 j_2 j_3; m'_1 m'_2, -m'_3) C^*(j_1 j_2 j_3; m''_1 m''_2, -m''_3) \\ = \int_0^\pi \sin\theta d\theta d^{j_1}_{m'_1 m''_1}(\theta) d^{j_2}_{m'_2 m''_2}(\theta) d^{j_3}_{m'_3 m''_3}(\theta). \end{aligned} \tag{402}$$

The d-functions are real. Thus, we can choose our arbitrary phase in C for given $j_1 j_2 j_3$ so that at least one non-zero coefficient is real. Then, according to this formula, all coefficients for given $j_1 j_2 j_3$ will be real. We therefore adopt the convention that all CG coefficients are real.

The right-hand side of Eqn. 402 is highly symmetric. Thus, it is useful to define the "3-j" symbol (Wigner):

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \frac{(-)^{j_1 - j_2 - m_3} C(j_1 j_2 j_3; m_1 m_2, -m_3)}{\sqrt{2j_3 + 1}}. \tag{403}$$

In terms of these 3-j symbols we have:

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m''_1 & m''_2 & m''_3 \end{pmatrix} \\ = \frac{1}{2} \int_0^\pi \sin\theta d\theta(\theta) d^{j_1}_{m'_1 m''_1}(\theta) d^{j_2}_{m'_2 m''_2}(\theta) d^{j_3}_{m'_3 m''_3}(\theta), \end{aligned} \tag{404}$$

for $m'_1 + m'_2 + m'_3 = m''_1 + m''_2 + m''_3 = 0$.

According to the symmetry of the right hand side, we see, for example, that the square of the 3-j symbol is invariant under permutations of the columns. Furthermore, since

$$d^j_{m'm''}(\pi - \theta) = (-)^{j - m''} d^j_{-m'm''}(\theta), \tag{405}$$

we find that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{406}$$

It is interesting to consider the question of which irreps occur in the symmetric, and in the anti-symmetric, tensor products of D^j with itself:

Theorem:

$$\left[D^j(u) \otimes D^j(u)\right]_s = \sum_{n=0}^{[j]} D^{2j-2n}(u), \tag{407}$$

$$\left[D^j(u) \otimes D^j(u)\right]_a = \sum_{n=0}^{[j-\frac{1}{2}]} D^{2j-2n-1}(u), \tag{408}$$

where s denotes the symmetric tensor product, a denotes the anti-symmetric tensor product, and $[j]$ means the greatest integer which is not larger than j.

Proof: We prove this by considering the characters. Let $S(u) \equiv [D^j(u) \otimes D^j(u)]_s$, and $A(u) \equiv [D^j(u) \otimes D^j(u)]_a$. Then, by definition,

$$S_{(m_1'm_2')(m_1''m_2'')} = \frac{1}{2}\left(D^j_{m_1'm_1''}D^j_{m_2'm_2''} + D^j_{m_1'm_2''}D^j_{m_2'm_1''}\right), \tag{409}$$

$$A_{(m_1'm_2')(m_1''m_2'')} = \frac{1}{2}\left(D^j_{m_1'm_1''}D^j_{m_2'm_2''} - D^j_{m_1'm_2''}D^j_{m_2'm_1''}\right). \tag{410}$$

$$\tag{411}$$

Taking the traces means to set $m_1' = m_1''$ and $m_2' = m_2''$, and sum over m_1' and m_2'. This yields

$$\mathrm{Tr}S(u) = \frac{1}{2}[\chi_j^2(u) + \chi_j(u^2)] \tag{412}$$

$$\mathrm{Tr}A(u) = \frac{1}{2}[\chi_j^2(u) - \chi_j(u^2)]. \tag{413}$$

We need to evaluate $\chi_j(u^2)$. If u is a rotation by θ, then u^2 is a rotation by 2θ, hence,

$$\chi_j(u^2) = \sum_{m=-j}^{j} e^{-2im\theta} \tag{414}$$

$$= \sum_{k=-2j}^{2j} e^{-ik\theta} - \sum_{k=-2j+1}^{2j-1} e^{-ik\theta} + \sum_{k=-2j+2}^{2j-2} e^{-ik\theta} - \dots \quad (415)$$

$$= \sum_{n=0}^{2j} (-)^n \sum_{k=-2j+n}^{2j-n} e^{-ik\theta} \quad (416)$$

$$= \sum_{n=0}^{2j} (-)^n \chi_{2j-n}(u). \quad (417)$$

Next, consider $\chi_j^2(u)$. Since

$$\chi_{j'}(u)\chi_{j''}(u) = \sum_{j=|j'-j''|}^{j'+j''} \chi_j(u), \quad (418)$$

we have

$$\chi_j^2(u) = \sum_{k=0}^{2j} \chi_k(u) = \sum_{n=0}^{2j} \chi_{2j-n}(u). \quad (419)$$

Thus,

$$\mathrm{Tr}S(u) = \frac{1}{2} \sum_{n=0}^{2j} \left[\chi_{2j-n}(u) + (-)^n \chi_{2j-n}(u)\right] \quad (420)$$

$$= \sum_{n=0}^{[j]} \chi_{2j-2n}(u). \quad (421)$$

Similarly, we obtain:

$$\mathrm{Tr}A(u) = \sum_{n=0}^{[j-\frac{1}{2}]} \chi_{2j-2n-1}(u). \quad (422)$$

This completes the proof.

This theorem implies an important symmetry relation for the CG coefficients when two j's are equal. From Eqn. 398, we obtain

$$C(j_1 j_1 j; m_1' m_2' m') C(j_1 j_1 j; m_1'' m_2'' m'') = (2j+1) \int_{SU(2)} d(u) D^{*j}_{m'm''}(u) D^{j_1}_{m_1' m_1''}(u) D^{j_1}_{m_2' m_2''}(u). \quad (423)$$

But

$$D^{j_1}_{m_1'm_1''}(u)D^{j_1}_{m_2'm_2''}(u) = S_{(m_1'm_2')(m_1''m_2'')}(u) + A_{(m_1'm_2')(m_1''m_2'')}(u), \tag{424}$$

and the integral of this with $D^{*j}(u)$ picks the D^j piece of this quantity, which must be either $m_1' \leftrightarrow m_2'$ symmetric or anti-symmetric, according to the theorem we have just proven. If symmetric, then

$$C(j_1j_1j;m_2'm_1'm') = C(j_1j_1j;m_1'm_2'm'), \tag{425}$$

and if anti-symmetric, then

$$C(j_1j_1j;m_2'm_1'm') = -C(j_1j_1j;m_1'm_2'm'). \tag{426}$$

Let's try a simple example: Let $j_1 = j_2 = \frac{1}{2}$. That is, we wish to combine two spin-1/2 systems (with zero orbital angular momentum). From the theorem:

$$\left(D^{\frac{1}{2}} \otimes D^{\frac{1}{2}}\right)_s = \sum_{n=0}^{[1/2]} D^{1-2n} = D^1, \tag{427}$$

$$\left(D^{\frac{1}{2}} \otimes D^{\frac{1}{2}}\right)_a = \sum_{n=0}^{[0]} D^{1-2n-1} = D^0. \tag{428}$$

Hence, the spin-1 combination is symmetric, with basis

$$|j=1, m=1; m_1 = \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle, \tag{429}$$

$$\frac{1}{\sqrt{2}}\left(|1,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle + |1,0;-\tfrac{1}{2},\tfrac{1}{2}\rangle\right), \tag{430}$$

$$|1,-1;-\tfrac{1}{2},-\tfrac{1}{2}\rangle, \tag{431}$$

and the spin-0 combination is antisymmetric:

$$\frac{1}{\sqrt{2}}\left(|0,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle - |0,0;-\tfrac{1}{2},\tfrac{1}{2}\rangle\right). \tag{432}$$

The generalization of this example is that the symmetric combinations are $j = 2j_1, 2j_1 - 2, \ldots$, and the antisymmetric combinations are $j = 2j_1 - 1, 2j_1 - 3, \ldots$ Therefore,

$$C(j_1j_1j;m_2'm_1'm') = (-)^{2j_1+j}C(j_1j_1j;m_1'm_2'm'). \tag{433}$$

Also,

$$\begin{pmatrix} j & j & J \\ m_1 & m_2 & M \end{pmatrix} = (-)^{2j+J} \begin{pmatrix} j & j & J \\ m_2 & m_1 & M \end{pmatrix}, \tag{434}$$

as well as the corresponding column permutations, *e.g.*,

$$\begin{pmatrix} j & J & j \\ m_1 & M & m_2 \end{pmatrix} = (-)^{2j+J} \begin{pmatrix} j & J & J \\ m_2 & M & m_1 \end{pmatrix}, \tag{435}$$

We adopt a "standard construction" of the 3-j and CG coefficients: First, they are selected to be real. Second, for any triplet $j_1 j_2 j$ the 3-j symbols are then uniquely determined, except for an overall sign which depends on $j_1 j_2 j$ only. By convention, we pick (this is a convention when j_1, j_2, j_3 are all different, otherwise it is required):

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = (-)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}. \tag{436}$$

That is, the 3-j symbol is chosen to be symmetric under cyclic permutation of the columns, and either symmetric or anti-symmetric, depending on $j_1 + j_2 + j_3$, under anti-cyclic permutations.

Sometimes the symmetry properties are all we need, *e.g.*, to determine whether some process is permitted or not by angular momentum conservation. However, we often need to know the CG (or 3-j) coefficients themselves. These are tabulated in many places. We can also compute them ourselves, and we now develop a general formula for doing this. We can take the following as the defining relation for the 3-j symbols, *i.e.*, it can be shown to be consistent with all of the above constraints:

$$\begin{aligned} G(\{k\};\{x\},\{y\}) &\equiv \frac{(x_1y_2 - x_2y_1)^{2k_3}(x_2y_3 - x_3y_2)^{2k_1}(x_3y_1 - x_1y_3)^{2k_2}}{\sqrt{(2k_3)!(2k_1)!(2k_2)!(j_1+j_2+j_3+1)!}} \\ &\equiv \sum_{m_1m_2m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} P_{j_1m_1}(x_1,y_1)P_{j_2m_2}(x_2,y_2)P_{j_3m_3}(x_3,y_3), \end{aligned} \tag{437}$$

where $2k_1, 2k_2, 2k_3$ are non-negative integers given by:

$$2k_3 = j_1 + j_2 - j_3; \quad 2k_1 = j_2 + j_3 - j_1; \quad 2k_2 = j_3 + j_1 - j_2. \tag{438}$$

We'll skip the proof of this consistency here. The interested reader may wish to look at T.Regge, *Il Nuovo Cimento* **X** (1958) 296; V. Bargmann, *Rev.*

Mod. Phys. **34** (1962) 829. Since $P_{jm}(\partial_x, \partial_y)P_{jm'}(x,y) = \delta_{mm'}$, we obtain the explicit formula:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = P_{j_1m_1}(\partial_{x_1}, \partial_{y_1})P_{j_2m_2}(\partial_{x_2}, \partial_{y_2})P_{j_3m_3}(\partial_{x_3}, \partial_{y_3})G(\{k\}; \{x\}, \{y\}). \tag{439}$$

For example consider the special case in which $j_3 = j_1 + j_2$ (and $m_3 = -(m_1 + m_2)$). In this case, $k_3 = 0$, $k_1 = j_2$, and $k_2 = j_1$. Thus,

$$\begin{pmatrix} j_1 & j_2 & j_1+j_2 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} \tag{440}$$

$$= \frac{\partial_{x_1}^{j_1+m_1}\partial_{y_1}^{j_1-m_1}\partial_{x_2}^{j_2+m_2}\partial_{y_2}^{j_2-m_2}\partial_{x_3}^{j_1+j_2-m_1-m_2}\partial_{y_3}^{j_1+j_2+m_1+m_2}}{\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_1+j_2-m_1-m_2)!(j_1+j_2+m_1+m_2)!}}$$

$$\times \frac{(x_2y_3 - x_3y_2)^{2j_2}(x_3y_1 - x_1y_3)^{2j_1}}{\sqrt{(2j_2)!(2j_1)!(2j_1+2j_2+1)!}} \tag{441}$$

$$= (-)^{j_1+j_2+m_1-m_2}\sqrt{\frac{(2j_1)!(2j_2)!(j_1+j_2-m_1-m_2)!(j_1+j_2+m_1+m_2)!}{(2j_1+2j_2+1)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}}. \tag{442}$$

For $j_1 = j_3 = j$, $j_2 = 0$, we find

$$\begin{pmatrix} j & 0 & j \\ m & 0 & -m \end{pmatrix} = \frac{(-)^{j+m}}{\sqrt{2j+1}}. \tag{443}$$

We may easily derive the corresponding formulas for the CG coefficients. In constructing tables of coefficients, much computation can be saved by using symmetry relations and orthogonality of states. For example, we have really already computed the table for the $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ case:

$C(\frac{1}{2}\frac{1}{2}j; m_1m_2m)$

		j	1	1	0	1
m_1	m_2	m	1	0	0	-1
$\frac{1}{2}$	$\frac{1}{2}$		1			
$\frac{1}{2}$	$-\frac{1}{2}$			$1/\sqrt{2}$	$1/\sqrt{2}$	
$-\frac{1}{2}$	$\frac{1}{2}$			$1/\sqrt{2}$	$-1/\sqrt{2}$	
$-\frac{1}{2}$	$-\frac{1}{2}$					1

For $\frac{1}{2} \otimes 1 = \frac{1}{2} \oplus \frac{3}{2}$ we find:

$C(1\tfrac{1}{2}j; m_1 m_2 m)$

		j	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
m_1	m_2	m	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
1	$\frac{1}{2}$		1					
1	$-\frac{1}{2}$			$1/\sqrt{3}$	$\sqrt{2/3}$			
0	$\frac{1}{2}$			$\sqrt{2/3}$	$-1/\sqrt{3}$			
0	$-\frac{1}{2}$					$\sqrt{2/3}$	$1/\sqrt{3}$	
-1	$\frac{1}{2}$					$1/\sqrt{3}$	$-\sqrt{2/3}$	
-1	$-\frac{1}{2}$							1

14 Wigner-Eckart Theorem

Consider a complex vector space $\mathcal{H}_{\text{op}}$ of operators on $\mathcal{H}$ which is closed under $U(u)QU^{-1}(u)$, where $Q \in \mathcal{H}_{\text{op}}$, and $U(u)$ is a continuous unitary representation of $SU(2)$. We denote the action of an element of the group $SU(2)$ on $Q \in \mathcal{H}_{\text{op}}$ by

$$\hat{U}(u)Q = U(u)QU^{-1}(u). \tag{444}$$

Corresponding to this action of the group elements, we have the action of the Lie algebra of $SU(2)$:

$$\hat{J}_k Q = [J_k, Q]. \tag{445}$$

We have obtained this result by noting that, picking a rotation about the k axis, with $U(u) = \exp(-i\theta J_k)$,

$$\begin{aligned} U(u)QU^{-1}(u) &= (1 - i\theta J_k)Q(1 + i\theta J_k) + O(\theta^2) &(446)\\ &= Q - i\theta[J_k, Q] + O(\theta^2), &(447) \end{aligned}$$

and comparing with

$$\hat{U}(u)Q = \exp(-i\theta \hat{J}_k)Q = Q - i\theta \hat{J}_k Q + O(\theta^2). \tag{448}$$

We may also compute the commutator:

$$\begin{aligned} [\hat{J}_k, \hat{J}_\ell]Q &= \hat{J}_k[J_\ell, Q] - \hat{J}_\ell[J_k, Q] &(449)\\ &= [J_k, [J_\ell, Q]] - [J_\ell, [J_k, Q]] &(450)\\ &= [[J_k, J_\ell], Q] &(451)\\ &= [i\epsilon_{k\ell m} J_m, Q]. &(452) \end{aligned}$$

Hence,

$$[\hat{J}_k, \hat{J}_\ell] = i\epsilon_{k\ell m} \hat{J}_m. \tag{453}$$

Def: A set of operators $Q(j, m)$, $m = -j, -j+1, \ldots, j$ consists of the $2j+1$ components of a **spherical tensor of rank** j if:

1. $\hat{J}_3 Q(j, m) = [J_3, Q(j, m)] = mQ(j, m)$.
2. $\hat{J}_+ Q(j, m) = [J_+, Q(j, m)] = \sqrt{(j - m)(j + m + 1)}Q(j, m + 1)$.

3. $\hat{J}_- Q(j,m) = [J_-, Q(j,m)] = \sqrt{(j+m)(j-m+1)}Q(j,m-1)$.

Equivalently,

$$\hat{\mathbf{J}}^2 Q(j,m) = \sum_{k=1}^{3} (\hat{J}_k)^2 Q(j,m) \quad (454)$$

$$= \sum_{k=1}^{3} [J_k, [J_k, Q(j,m)]] \quad (455)$$

$$= j(j+1)Q(j,m), \quad (456)$$

and

$$U(u)Q(j,m)U^{-1}(u) = \sum_{m'} D^j_{m'm}(u)Q(j,m'). \quad (457)$$

That is, the set of operators $Q(j,m)$ forms a spherical tensor if they transform under rotations like the basis vectors $|jm\rangle$ in the standard representation.

For such an object, we conclude that the matrix elements of $Q(j,m)$ must depend on the m values in a particular way (letting k denote any "other" quantum numbers describing our state in the situation):

$$\langle (j'm')(k')|Q(j,m)|(j''m'')(k'')\rangle \quad (458)$$

$$= \text{“}\langle j'm'|(|jm\rangle|j''m''\rangle)\langle j',k'||Q_j||j'',k''\rangle\text{”} \quad (459)$$

$$= A(j,j',j'')C(jj''j';mm''m')\langle j',k'||Q_j||j'',k''\rangle, \quad (460)$$

where a common convention lets

$$A(j,j',j'') = (-)^{j+j'-j''}/\sqrt{2j'+1}. \quad (461)$$

The symbol $\langle j',k'||Q_j||j'',k''\rangle$ is called the **reduced matrix element** of the tensor Q_j. Eqn. 460 is the statement of the "Wigner-Eckart theorem".

Let us try some examples:

1. Scalar operator: In the case of a scalar operator, there is only one component:

$$Q(j,m) = Q(0,0). \quad (462)$$

The Wigner-Eckart theorem reads

$$\langle (j'm')(k')|Q(0,0)|(j''m'')(k'')\rangle \quad (463)$$

$$= \frac{(-)^{j'-j''}}{\sqrt{2j'+1}} C(0j''j';0m''m')\langle j',k'||Q_0||j'',k''\rangle. \quad (464)$$

But

$$
\begin{aligned}
C(0j''j';0m''m') &= \langle j'm'|(|00\rangle|j''m''\rangle) & (465)\\
&= \delta_{j'j''}\delta_{m'm''}, & (466)
\end{aligned}
$$

and hence,

$$
\langle(j'm')(k')|Q(0,0)|(j''m'')(k'')\rangle = \delta_{j'j''}\delta_{m'm''}\frac{\langle j',k'||Q_0||j'',k''\rangle}{\sqrt{2j'+1}}. \quad (467)
$$

The presence of the kronecker deltas tells us that a scalar operator cannot change the angular momentum of a system, *i.e.*, the matrix element of the operator between states of differing angular momenta is zero.

2. Vector operator: For $j=1$ the Wigner-Eckart theorem is:

$$
\begin{aligned}
&\langle(j'm')(k')|Q(1,m)|(j''m'')(k'')\rangle & (468)\\
&= \frac{(-)^{1+j'-j''}}{\sqrt{2j'+1}}C(1j''j';mm''m')\langle j',k'||Q_1||j'',k''\rangle. & (469)
\end{aligned}
$$

Before pursuing this equation with an example, let's consider the construction of the tensor components of a vector operator. We are given, say, the Cartesian components of the operator: $\mathbf{Q}=(Q_x,Q_y,Q_z)$. We wish to find the tensor components $Q(1,-1),Q(1,0),Q(1,1)$ in terms of these Cartesian components, in the standard representation. We must have:

$$
\begin{aligned}
\hat{J}_3 Q(j,m) &= [J_3,Q(j,m)] = mQ(j,m), & (470)\\
\hat{J}_\pm Q(j,m) &= [J_\pm,Q(j,m)] = \sqrt{(j\mp m)(j\pm m+1)}Q(j,m\pm 1). & (471)
\end{aligned}
$$

The Q_x,Q_y,Q_z components of a vector operator obey the commutation relations with angular momentum:

$$
[J_k,Q_\ell] = i\epsilon_{k\ell m}Q_m. \quad (472)
$$

Thus, consistency with the desired relations is obtained if

$$
\begin{aligned}
Q(1,1) &= -\frac{1}{\sqrt{2}}(Q_x+iQ_y) & (473)\\
Q(1,0) &= Q_z & (474)\\
Q(1,-1) &= \frac{1}{\sqrt{2}}(Q_x-iQ_y). & (475)
\end{aligned}
$$

These, then, are the components of a spherical tensor of rank 1, expressed in terms of Cartesian components.

Now let's consider the case where **Q**=**J**, that is, the case in which our vector operator is the angular momentum operator. To evaluate the reduced matrix element, we chose any convenient component, for example, $Q(1,0) = J_z$. Hence,

$$\begin{aligned} \langle j'm'(k')|J_z|j''m''(k'')\rangle &= \delta_{j'j''}\delta_{m'm''}\delta_{k'k''}m'' \qquad (476) \\ &= \frac{\langle j',k'||J||j'',k''\rangle}{\sqrt{2j'+1}} C(1j''j';0m''m')(-)^{1+j'+j''}. \qquad (477) \end{aligned}$$

We see that the reduced matrix element vanishes unless $j' = j''$ (and $k' = k''$).

The relevant CG coefficients are given by

$$C(1j'j';0m'm') = \sqrt{2j'+1}(-)^{1-j'+m'} \begin{pmatrix} 1 & j' & j' \\ 0 & m' & -m' \end{pmatrix}, \qquad (478)$$

where

$$\begin{aligned} &\begin{pmatrix} 1 & j' & j' \\ 0 & m' & -m' \end{pmatrix} \qquad (479) \\ &= P_{10}(\partial_{x_1},\partial_{y_1})P_{j'm'}(\partial_{x_2},\partial_{y_2})P_{j',-m'}(\partial_{x_3},\partial_{y_3})G(\{k\};\{x\},\{y\}) \qquad (480) \\ &= (-)^{j'-m'}\frac{2m'}{\sqrt{(2j'+2)(2j'+1)2j'}}, \qquad (481) \end{aligned}$$

which the reader is invited to demonstrate (with straightforward, though somewhat tedious, algebra). Therefore,

$$C(1j'j';0m'm') = -\frac{m'}{\sqrt{j'(j'+1)}}. \qquad (482)$$

Inserting the CG coefficients into our expression for the reduced matrix element, we find

$$\frac{\langle j',k'||J||j'',k''\rangle}{\sqrt{2j'+1}} = \delta_{j'j''}\delta_{k'k''}\sqrt{j'(j'+1)}. \qquad (483)$$

We see that this expression behaves like $\sqrt{\langle \mathbf{J}^2 \rangle}$. Plugging back into the Wigner-Eckart theorem, we find:

$$\langle (j'm')(k')|J_m|(j''m'')(k'')\rangle \tag{484}$$
$$= \delta_{j'j''}\delta_{k'k''}\sqrt{j'(j'+1)}[-C(1j'j';mm''m')], \tag{485}$$

where $J_m = J_1, J_0, J_{-1}$ denotes the tensor components of $\mathbf{J}$.

Consider, for example, $J(1,1) = -\frac{1}{\sqrt{2}}J_+$. The Wigner-Eckart theorem now tells us that

$$\langle (jm+1| - \frac{1}{\sqrt{2}}J_+|(jm)\rangle = -\frac{1}{\sqrt{2}}\sqrt{(j-m)(j+m+1)}, \tag{486}$$
$$= \sqrt{j(j+1)}[-C(1jj;1mm+1)] \tag{487}$$

Thus, we have found an expression which may be employed to compute some CG coefficients:

$$C(1jj;1mm+1) = \sqrt{\frac{(j-m)(j+m+1)}{2j(j+1)}}. \tag{488}$$

We see that the Wigner-Eckart theorem applied to J itself can be used to determine CG coefficients.

15 Exercises

1. Prove the theorem we state in this note:

Theorem: The most general mapping $\mathbf{x} \rightarrow \mathbf{x}'$ of R^3 into itself, such that the origin is mapped into the origin, and such that all distances are preserved, is a linear, real orthogonal transformation Q:

$$\mathbf{x}' = Q\mathbf{x}, \quad \text{where} \quad Q^TQ = I, \quad \text{and} \quad Q^* = Q. \tag{489}$$

Hence,

$$\mathbf{x}' \cdot \mathbf{y}' = \mathbf{x} \cdot \mathbf{y} \quad \forall \, \text{points} \, \mathbf{x}, \mathbf{y} \in R^3. \tag{490}$$

For such a mapping, either:

(a) $\det(Q) = 1$, Q is called a **proper** orthogonal transformation, and is in fact a rotation. In this case,

$$\boldsymbol{x}' \times \boldsymbol{y}' = (\boldsymbol{x} \times \boldsymbol{y})' \quad \forall \, \text{points} \, \boldsymbol{x}, \boldsymbol{y} \in R^3. \tag{491}$$

or,

(b) $\det(Q) = -1$, Q is called an **improper** orthogonal transformation, and is the product of a reflection (parity) and a rotation. In this case,

$$\boldsymbol{x}' \times \boldsymbol{y}' = -(\boldsymbol{x} \times \boldsymbol{y})' \quad \forall \, \text{points} \, \boldsymbol{x}, \boldsymbol{y} \in R^3. \tag{492}$$

The set of all orthogonal transformations forms a group (denoted $O(3)$), and the set of all proper orthogonal transformations forms a subgroup ($O^+(3)$ or $SO(3)$ of $O(3)$), identical with the set of all rotations.

[You may wish to make use of the following intuitive lemma: Let $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3$ be any three mutually perpendicular unit vectors such that:

$$\boldsymbol{e}'_3 = \boldsymbol{e}'_1 \times \boldsymbol{e}'_2 \quad \text{(right-handed system)}. \tag{493}$$

Then there exists a unique rotation $R_{\boldsymbol{u}}(\theta)$ such that

$$\boldsymbol{e}'_i = R_{\boldsymbol{u}}(\theta)\boldsymbol{e}_i, \quad i = 1, 2, 3. \tag{494}$$

]

2. We stated the following generalization of the addition law for tangents:

Theorem: If $R_{\boldsymbol{e}}(\theta) = R_{\boldsymbol{e}''}(\theta'')R_{\boldsymbol{e}'}(\theta')$, and defining:

$$\boldsymbol{\tau} = \boldsymbol{e} \tan\theta/2 \tag{495}$$
$$\boldsymbol{\tau}' = \boldsymbol{e}' \tan\theta'/2 \tag{496}$$
$$\boldsymbol{\tau}'' = \boldsymbol{e}'' \tan\theta''/2, \tag{497}$$

then:

$$\boldsymbol{\tau} = \frac{\boldsymbol{\tau}' + \boldsymbol{\tau}'' + \boldsymbol{\tau}'' \times \boldsymbol{\tau}'}{1 - \boldsymbol{\tau}' \cdot \boldsymbol{\tau}''}. \tag{498}$$

A simple way to prove this theorem is to use SU(2) to represent the rotations, *i.e.*, the rotation $R_{\boldsymbol{e}}(\theta)$ is represented by the SU(2) matrix $\exp\left(-\frac{i}{2}\theta\boldsymbol{e}\cdot\sigma\right)$. You are asked to carry out this proof.

3. We made the assertion that if we had an element $u \in SU(2)$ which commuted with every element of the vector space of traceless 2×2 Hermitian matrices, then u must be a multiple of the identity (*i.e.*, either $u = I$ or $u = -I$). Let us demonstrate this, learning a little more group theory along the way.

   First, we note that if we have a matrix group, it is possible to generate another matrix representation of the group by replacing each element with another according to the mapping:

$$u \rightarrow v \tag{499}$$

   where

$$v = SuS^{-1}. \tag{500}$$

   and S is any chosen non-singular matrix.

   (a) Show that if $\{u\}$ is a matrix group, then $\{v|v = SuS^{-1};\ S$ a non-singular matrix$\}$ is a representation of the group (*i.e.*, the mapping is 1 : 1 and the multiplication table is preserved under the mapping). The representations $\{u\}$ and $\{v\}$ are considered to be equivalent.

   A group of unitary matrices is said to be **reducible** if there exists a mapping of the above form such that every element may simultaneously be written in block-diagonal form:

$$M(g) = \begin{pmatrix} A(g) & 0 & 0 \\ 0 & B(g) & 0 \\ 0 & 0 & \ddots \end{pmatrix}$$

   $\forall g \in$ the group ($A(g)$ and $B(g)$ are sub-matrices).

   (b) Show that $SU(2)$ is not reducible (*i.e.*, $SU(2)$ is irreducible).

   (c) Now prove the following useful lemma: A matrix which commutes with every element of an irreducible matrix group is a multiple of the identity matrix. [Hint: Let B be such a matrix commuting with every element, and consider the eigenvector equation $B\mathbf{x} =$

$\lambda\mathbf{x}$. Then consider the vector $u\mathbf{x}$ where u is any element of the group, and $\mathbf{x}$ is the eigenvector corresponding to eigenvalue λ.]

(d) Finally, prove the assertion we stated at the beginning of this problem.

4. We have discussed rotations using the language of group theory. Let us look at a simple application of group theory in determining "selection rules" implied by symmetry under the elements of the group (where the group is a group of operations, such as rotations). The point is that we can often predict much about the physics of a situation simply by "symmetry" arguments, without resorting to a detailed solution.

   Consider a positronium "atom", *i.e.*, the bound state of an electron and a positron. The relevant binding interaction here is electromagnetism. The electromagnetic interaction doesn't depend on the orientation of the system, that is, it is invariant with respect to rotations. It also is invariant with respect to reflections. You may wish to convince yourself of these statements by writing down an explicit Hamiltonian, and verifying the invariance.

   Thus, the Hamiltonian for positronium is invariant with respect to the group $O(3)$, and hence commutes with any element of this group. Hence, angular momentum and parity are conserved, and the eigenstates of energy are also eigenstates of parity and total angular momentum (J). In fact, the spin and orbital angular momentum degrees of freedom are sufficiently decoupled that the total spin (S) and orbital angular momentum (L) are also good quantum numbers for the energy eigenstates to an excellent approximation. The ground state of positronium ("parapositronium") is the ${}^1S_0$ state in ${}^{2S+1}L_J$ spectroscopic notation, where $L = S$ means zero orbital angular momentum. Note that the letter "S" in the spectroscopic notation is not the same as the "S" referring to the total spin quantum number. Sorry about the confusion, but it's established tradition...

   In the ground state, the positron and electron have no relative orbital angular momentum, and their spins are anti-parallel. The first excited state ("orthopositronium") is the ${}^3S_1$ state, in which the spins of the positron and electron are now aligned parallel with each other. The ${}^3S_1 -{}^1 S_0$ splitting is very small, and is analogous to the "hyperfine" splitting in normal atoms.

Positronium decays when the electron and positron annihilate to produce photons. The decay process is also electromagnetic, hence also governed by a Hamiltonian which is invariant under $O(3)$. As a consequence of this symmetry, angular momentum and parity are conserved in the decay.

(a) We said that parity was a good quantum number for positronium states. To say just what the parity is, we need to anticipate a result from the Dirac equation (sorry): The intrinsic parities of the electron and positron are opposite. What is the parity of parapositronium? Of orthopositronium?

(b) We wish to know whether positronium can decay into two photons. Let us check parity conservation. What are the possible parities of a state of two photons, in the center-of-mass frame? Can you exclude the decay of positronium to two photons on the basis of parity conservation?

(c) Let us consider now whether rotational invariance, *i.e.*, conservation of angular momentum, puts any constraints on the permitted decays of positronium. Can the orthopositronium state decay to two photons? What about the parapositronium state?

5. The "charge conjugation" operator, C, is an operator that changes all particles into their anti-particles. Consider the group of order 2 generated by the charge conjugation operator. This group has elements $\{I, C\}$, where I is the identity element. The electromagnetic interaction is invariant with respect to the actions of this group. That is, any electromagnetic process for a system of particles should proceed identically if all the particles are replaced by their anti-particles. Hence, C is a conserved quantity. Let's consider the implications of this for the $^1S_0$ and $^3S_1$ positronium decays to two photons. [See the preceeding exercise for a discussion of positronium. Note that you needn't have done that problem in order to do this one.]

   (a) The result of operating C on a photon is to give a photon, *i.e.*, the photon is its own anti-particle, and is thus an eigenstate of C. What is the eigenvalue? That is, what is the "C-parity" of the photon? You should give your reasoning. No credit will be given for just writing down the answer, even if correct. [Hint:

think classically about electromagnetic fields and how they are produced.] Hence, what is the C-parity of a system of n photons?

(b) It is a bit trickier to figure out the charge conjugation of the positronium states. Since these are states consisting of a particle and its antiparticle, we suspect that they may also be eigenstates of C. But is the eigenvalue positive or negative? To determine this, we need to know a bit more than we know so far.

Let me give an heuristic argument for the new understanding that we need. First, although we haven't talked about it yet, you probably already know about the "Pauli Exclusion Principle", which states that two identical fermions cannot be in the same state.

Suppose we have a state consisting of two electrons, $|\boldsymbol{x}_1, \boldsymbol{S}_1; \boldsymbol{x}_2, \boldsymbol{S}_2\rangle$. We may borrow an idea we introduced in our discussion of the harmonic oscillator, and define a "creation operator", $a^\dagger(\boldsymbol{x}, \boldsymbol{S})$, which creates an electron at $\boldsymbol{x}$ with spin $\boldsymbol{S}$. Consider the two-electron state:

$$[a^\dagger(\boldsymbol{x}_1, \boldsymbol{S}_1)a^\dagger(\boldsymbol{x}_2, \boldsymbol{S}_2) + a^\dagger(\boldsymbol{x}_2, \boldsymbol{S}_2)a^\dagger(\boldsymbol{x}_1, \boldsymbol{S}_1)]|0\rangle, \tag{501}$$

where $|0\rangle$ is the "vacuum" state, with no electrons. But this puts both electrons in the same state, since it is invariant under the interchange $1 \leftrightarrow 2$. Therefore, in order to satisfy the Pauli principle, we must have that

$$a^\dagger(\boldsymbol{x}_1, \boldsymbol{S}_1)a^\dagger(\boldsymbol{x}_2, \boldsymbol{S}_2) + a^\dagger(\boldsymbol{x}_2, \boldsymbol{S}_2)a^\dagger(\boldsymbol{x}_1, \boldsymbol{S}_1) = 0 \tag{502}$$

That is, the creation operators anti-commute. To put it another way, if two electrons are interchanged, a minus sign is introduced. You may be concerned that a positron and an electron are non-identical particles, so maybe this has nothing to do with positronium. However, the relativistic description is such that the positron and electron may be regarded as different "components" of the electron (*e.g.*, the positron may be interpreted in terms of "negative-energy" electron states), so this anti-commutation relation is preserved even when creating electrons and positrons.

Determine the C-parity of the $^3S_1$ and $^1S_0$ states of positronium, and thus deduce whether decays to two photons are permitted according to conservation of C. [Hint: Consider a positronium state and let C act on it. Relate this back to the original state by appropriate transformations.]

6. Suppose we have a system with total angular momentum 1. We pick a basis corresponding to the three eigenvectors of the z-component of angular momentum, J_z, with eigenvalues $+1, 0, -1$, respectively. We are given an ensemble described by density matrix:

$$\rho = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

   (a) Is ρ a permissible density matrix? Give your reasoning. For the remainder of this problem, assume that it is permissible. Does it describe a pure or mixed state? Give your reasoning.

   (b) Given the ensemble described by ρ, what is the average value of J_z?

   (c) What is the spread (standard deviation) in measured values of J_z?

7. Let us consider the application of the density matrix formalism to the problem of a spin-1/2 particle (such as an electron) in a static external magnetic field. In general, a particle with spin may carry a magnetic moment, oriented along the spin direction (by symmetry). For spin-1/2, we have that the magnetic moment (operator) is thus of the form:

$$\boldsymbol{\mu} = \frac{1}{2}\gamma\boldsymbol{\sigma}, \tag{503}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices, the $\frac{1}{2}$ is by convention, and γ is a constant, giving the strength of the moment, called the gyromagnetic ratio. The term in the Hamiltonian for such a magnetic moment in an external magnetic field, $\boldsymbol{B}$ is just:

$$H = -\boldsymbol{\mu} \cdot \boldsymbol{B}. \tag{504}$$

Our spin-1/2 particle may have some spin-orientation, or "polarization vector", given by:

$$\boldsymbol{P} = \langle \boldsymbol{\sigma} \rangle. \tag{505}$$

Drawing from our classical intuition, we might expect that in the external magnetic field the polarization vector will exhibit a precession about the field direction. Let us investigate this.

Recall that the expectation value of an operator may be computed from the density matrix according to:

$$\langle A \rangle = \mathrm{Tr}(\rho A). \tag{506}$$

Furthermore, recall that the time evolution of the density matrix is given by:

$$i\frac{\partial \rho}{\partial t} = [H(t), \rho(t)]. \tag{507}$$

What is the time evolution, $d\mathbf{P}/dt$, of the polarization vector? Express your answer as simply as you can (more credit will be given for right answers that are more physically transparent than for right answers which are not). Note that we make no assumption concerning the purity of the state.

8. Let us consider a system of N spin-1/2 particles (as in the previous problem) per unit volume in thermal equilibrium, in our external magnetic field $\mathbf{B}$. [Even though we refer to the previous exercise, the solution to this problem does not require solving the previous one.] Recall that the canonical distribution is:

$$\rho = \frac{e^{-H/T}}{Z}, \tag{508}$$

with partition function:

$$Z = \mathrm{Tr}\left(e^{-H/T}\right). \tag{509}$$

Such a system of particles will tend to orient along the magnetic field, resulting in a bulk magnetization (having units of magnetic moment per unit volume), $\mathbf{M}$.

   (a) Give an expression for this magnetization (don't work too hard to evaluate).

   (b) What is the magnetization in the high-temperature limit, to lowest non-trivial order (this I want you to evaluate as completely as you can!)?

9. We have discussed Lie algrebras (with Lie product given by the commutator) and Lie groups, in our attempt to deal with rotations. At one point, we asserted that the structure (multiplication table) of the Lie group in some neighborhood of the identity was completely determined by the structure (multiplication table) of the Lie algebra. We noted that, however intuitively pleasing this might sound, it was not actually a trivial statement, and that it followed from the "Baker-Campbell-Hausdorff" theorem. Let's try to tidy this up a bit further here.

   First, let's set up some notation: Let $\mathcal{L}$ be a Lie algebra, and $\mathcal{G}$ be the Lie group generated by this algebra. Let $X, Y \in \mathcal{L}$ be two elements of the algebra. These generate the elements $e^X, e^Y \in \mathcal{G}$ of the Lie group. We assume the notion that if X and Y are close to the zero element of the Lie algebra, then e^X and e^Y will be close to the identity element of the Lie group.

   What we want to show is that the group product $e^X e^Y$ may be expressed in the form e^Z, where $Z \in \mathcal{L}$, at least for X and Y not too "large". Note that the non-trivial aspect of this problem is that, first, X and Y may not commute, and second, objects of the form XY may not be in the Lie algebra. Elements of $\mathcal{L}$ generated by X and Y must be linear combinations of X, Y, and their repeated commutators.

   (a) Suppose X and Y commute. Show explicitly that the product $e^X e^Y$ is of the form e^Z, where Z is an element of $\mathcal{L}$. (If you think this is trivial, don't worry, it is!)

   (b) Now suppose that X and Y may not commute, but that they are very close to the zero element. Keeping terms to quadratic order in X and Y, show once again that the product $e^X e^Y$ is of the form e^Z, where Z is an element of $\mathcal{L}$. Give an explicit expression for Z.

   (c) Finally, for more of a challenge, let's do the general theorem: Show that $e^X e^Y$ is of the form e^Z, where Z is an element of $\mathcal{L}$, as long as X and Y are sufficiently "small". We won't concern ourselves here with how "small" X and Y need to be – you may investigate that at more leisure.

   Here are some hints that may help you: First, we remark that the differential equation

$$\frac{df}{du} = Xf(u) + g(u), \tag{510}$$

where $X \in \mathcal{L}$, and letting $f(0) = f_0$, has the solution:

$$f(u) = e^{uX} f_0 + \int_0^u e^{(u-v)X} g(v) dv. \tag{511}$$

This can be readily verified by back-substitution. If g is independent of u, then the integral may be performed, with the result:

$$f(u) = e^{uX} f_0 + h(u, X) g, \tag{512}$$

Where, formally,

$$h(u, X) = \frac{e^{uX} - 1}{X}. \tag{513}$$

Second, if $X, Y \in \mathcal{L}$, then

$$e^X Y e^{-X} = e^{X_c}(Y), \tag{514}$$

where I have introduced the notation "X_c" to mean "take the commutator". That is, $X_c(Y) \equiv [X, Y]$. This fact may be demonstrated by taking the derivative of

$$A(u; Y) \equiv e^{uX} Y e^{-uX} \tag{515}$$

with respect to u, and comparing with our differential equation above to obtain the desired result.

Third, assuming $X = X(u)$ is differentiable, we have

$$e^{X(u)} \frac{d}{du} e^{-X(u)} = -h(1, X(u)_c) \frac{dX}{du}. \tag{516}$$

This fact may be verified by considering the object:

$$B(t, u) \equiv e^{tX(u)} \frac{\partial}{\partial u} e^{-tX(u)}, \tag{517}$$

and differentiating (carefully!) with respect to t, using the above two facts, and finally letting $t = 1$.

One final hint: Consider the quantity

$$Z(u) = \ln\left(e^{uX} e^Y\right). \tag{518}$$

The series:

$$\ell(z) = \frac{\ln z}{z - 1} = 1 - \frac{z-1}{2} + \frac{(z-1)^2}{3} - \cdots \tag{519}$$

plays a role in the explicit form for the result. Again, you are not asked to worry about convergence issues.

10. In an earlier exercise we considered the implication of rotational invariance for the decays of positronium states into two photons. Let us generalize and broaden that discussion here. Certain neutral particles (*e.g.*, π^0, η, η') are observed to decay into two photons, and others (*e.g.*, ω, ϕ, ψ) are not. Let us investigate the selection rules implied by angular momentum and parity conservation (satisfied by electromagnetic and strong interactions) for the decay of a particle (call it "X") into two photons. Thus, we ask the question, what angular momentum J and parity P states are allowed for two photons?

    Set up the problem in the center-of-mass frame of the two photons, with the z-axis in the direction of one photon. We know that since a photon is a massless spin-one particle, it has two possible spin states, which we can describe by its "helicity", *i.e.*, its spin projection along its direction of motion, which can take on the values ± 1. Thus, a system of two photons can have the spin states:

$$|\uparrow\uparrow\rangle,\ \ |\downarrow\downarrow\rangle,\ \ |\uparrow\downarrow + \downarrow\uparrow\rangle,\ \ |\uparrow\downarrow - \downarrow\uparrow\rangle$$

    (The first arrow refers to the photon in the $+z$ direction, the second to the photon in the $-z$ direction, and the direction of the arrow indicates the spin component along the z-axis, NOT to the helicity.) We consider the effect on these states of three operations (which, by parity and angular momentum conservation, should commute with the Hamiltonian):

    - P: parity – reverses direction of motion of a particle, but leaves its angular momentum unaltered.
    - $R_z(\alpha)$: rotation by angle α about the z-axis. A state with a given value of J_z (z-component of angular momentum) is an eigenstate, with eigenvalue $e^{i\alpha J_z}$.
    - $R_x(\pi)$: rotation by π about x-axis. For our two photons, this reverses the direction of motion and also the angular momentum of each photon. For our "X" particle, this operation has the effect corresponding to the effect on the spherical harmonic with the appropriate eigenvalues:

$$R_x(\pi)Y_{JJ_z}(\Omega)$$

(Note that the Y_{lm} functions are sufficient, since a fermion obviously can't decay into two photons and conserve angular momentum – hence X is a boson, and we needn't consider $\frac{1}{2}$-integer spins.)

Make sure that the above statements are intuitively clear to you.

(a) By considering the actions of these operations on our two-photon states, complete the following table: (one entry is filled in for you)

Photonic Spin State	Transformation P	$R_z(\alpha)$	$R_x(\pi)$
$\|\uparrow\uparrow\rangle$	$+\|\uparrow\uparrow\rangle$		
$\|\downarrow\downarrow\rangle$			
$\|\uparrow\downarrow + \downarrow\uparrow\rangle$			
$\|\uparrow\downarrow - \downarrow\uparrow\rangle$			

(b) Now fill in a table of eigenvalues for a state (*i.e.*, our particle "X") of arbitrary integer spin J and parity P (or, if states are not eigenvectors, what the transformations yield):

Spin J	Transformation P	$R_z(\alpha)$	$R_x(\pi)$
0	$\left\{ \begin{matrix} +1 \\ -1 \end{matrix} \right\}$		
1	$\left\{ \begin{matrix} +1 \\ -1 \end{matrix} \right\}$		
$2, 4, 6, \ldots$	$\left\{ \begin{matrix} +1 \\ -1 \end{matrix} \right\}$		
$3, 5, 7, \ldots$	$\left\{ \begin{matrix} +1 \\ -1 \end{matrix} \right\}$		

Note that there may be more than one eigenvalue of $R_z(\alpha)$ for a given row, corresponding to the different possible values of J_z.

(c) Finally, by using your answers to parts (a) and (b), determine the allowed and forbidden J^P states decaying into two photons, and the appropriate photonic helicity states for the allowed transitions. Put your answer in the form of a table:

Parity	Spin 0	1	2,4,...	3,5,...
+1				
−1	$\|\uparrow\downarrow - \downarrow\uparrow\rangle$			

You have (I hope) just derived something which is often referred to as "Yang's theorem". Note: People often get this wrong, so be careful!

11. We said that if we are given an arbitrary representation, $D(u)$, of $SU(2)$, it may be reduced to a direct sum of irreps $D^r(u)$:

$$D(u) = \sum_r \oplus D^r(u). \tag{520}$$

The **multiplicities** m_j (the number of irreducible representations D^r which belong to the equivalence class of irreducible representations characterized by index j) are unique, and they are given by:

$$m_j = \int_{SU(2)} d(u)\chi^j(u^{-1})\chi(u), \tag{521}$$

where $\chi(u) = \mathrm{Tr}\,[D(u)]$.

(a) Suppose you are given a representation, with characters:

$$\chi\left[u_{\mathbf{e}}(\theta)\right] = 1 + \frac{\sin\frac{3}{2}\theta + 2\sin\frac{7}{4}\theta\cos\frac{1}{4}\theta}{\sin\frac{1}{2}\theta}. \tag{522}$$

What irreducible representations appear in the reduction of this representation, with what multiplicities?

(b) Does the representation we are given look like it could correspond to rotations of a physically realizable system? Discuss.

12. In nuclear physics, we have the notion of "charge independence", or the idea that the nuclear (strong) force does not depend on whether we are dealing with neutrons or protons. Thus, the nuclear force is supposed to be symmetric with respect to unitary transformations on a space with basis vectors $p = \binom{1}{0}$, $n = \binom{0}{1}$. Neglecting the overall phase symmetry, we see that the symmetry group is SU(2), just as for rotations. As with angular momentum, we can generate representations of other dimensions, and work with systems of more than one nucleon. By analogy with angular momentum, we say that the neutron and proton form an "isotopic spin" (or "isospin") doublet, with

$$|n\rangle = |I = \frac{1}{2}, I_3 = -\frac{1}{2}\rangle \tag{523}$$

$$|p\rangle = |I = \frac{1}{2}, I_3 = +\frac{1}{2}\rangle \tag{524}$$

(The symbol "T" is also often used for isospin). Everything you know about SU(2) can now be applied in isotopic spin space.

Study the isobar level diagram of the He^6, Li^6, Be^6 nuclear level schemes, and discuss in detail the evidence for charge independence of the nuclear force. These graphs are quantitative, so your discussion should also be quantitative. Also, since these are real-life physical systems, you should worry about real-life effects which can modify an idealized vision.

You may find an appropriate level scheme via a google search (you want a level diagram for the nuclear isobars of 6 nucleons), *e.g.*, at:
http://www.tunl.duke.edu/nucldata/figures/06figs/06_is.pdf
For additional reference, you might find it of interest to look up:
F. Ajzenberg-Selove, "Energy Levels of Light Nuclei, A = 5-10," *Nucl. Phys.* **A490** 1-225 (1988)
(see also http://www.tunl.duke.edu/nucldata/fas/88AJ01.shtml).

13. We defined the "little-d" functions according to:

$$d^j_{m_1m_2}(\theta) = D^j_{m_1m_2}(0,\theta,0) = \langle j,m_1|e^{-i\theta J_2}|j,m_2\rangle$$

where the matrix elements $D^j_{m_1m_2}(\psi,\theta,\phi)$, parameterized by Euler angles ψ,θ,ϕ, are given in the "standard representation" by:

$$D^j_{m_1m_2}(\psi,\theta,\phi) = \langle j,m_1|D^j(u)|j,m_2\rangle = e^{-i(m_1\psi+m_2\phi)}d^j_{m_1m_2}(\theta)$$

We note that an explicit calculation for these matrix elements is possible via:

$$D^j_{m_1m_2}(u) = P_{jm_1}(\partial_x,\partial_y)P_{jm_2}(u_{11}x+u_{21}y, u_{12}x+u_{22}y) \quad (525)$$

where

$$P_{jm}(x,y) \equiv \frac{x^{j+m}y^{j-m}}{\sqrt{(j+m)!(j-m)!}}. \quad (526)$$

Prove the following handy formulas for the $d^j_{m_1m_2}(\theta)$ functions:

a) $\quad d^{j*}_{m_1m_2}(\theta) = d^j_{m_1m_2}(\theta) \qquad$ (reality of d^j functions)

b) $$d^j_{m_1m_2}(-\theta) = d^j_{m_2m_1}(\theta)$$

c) $$d^j_{m_1m_2}(\theta) = (-)^{m_1-m_2} d^j_{m_2m_1}(\theta)$$

d) $$d^j_{-m_1,-m_2}(\theta) = (-)^{m_1-m_2} d^j_{m_1m_2}(\theta)$$

e) $$d^j_{m_1m_2}(\pi-\theta) = (-)^{j-m_2} d^j_{-m_1,m_2}(\theta) = (-)^{j+m_1} d^j_{m_1,-m_2}(\theta)$$

f) $$d^j_{m_1m_2}(2\pi+\theta) = (-)^{2j} d^j_{m_1m_2}(\theta)$$

14. We would like to consider the (qualitative) effects on the energy levels of an atom which is moved from freedom to an external potential (a crystal, say) with cubic symmetry. Let us consider a one-electron atom and ignore spin for simplicity. Recall that the wave function for the case of the free atom looks something like $R_{nl}(r)Y_{lm}(\theta,\phi)$, and that all states with the same n and l quantum numbers have the same energy, *i.e.*, are $(2l+1)$-fold degenerate. The Hamiltonian for a free atom must have the symmetry of the full rotation group, as there are no special directions. Thus, we recall some properties of this group for the present discussion. First, we remark that the set of functions $\{Y_{lm} : m = -l, -l+1, \cdots, l-1, l\}$ for a given l forms the basis for a $(2l+l)$-dimensional subspace which is invariant under the operations of the full rotation group. [A set $\{\psi_i\}$ of vectors is said to span an *invariant subspace* V_s under a given set of operations $\{P_j\}$ if $P_j\psi_i \in V_s$ $\forall i, j$.] Furthermore, this subspace is "irreducible," that is, it cannot be split into smaller subspaces which are also invariant under the rotation group.

    Let us denote the linear transformation operator corresponding to element R of the rotation group by the symbol $\hat{P}_R$, *i.e.*:

$$\hat{P}_R f(\vec{x}) = f(R^{-1}\vec{x})$$

The way to think about this equation is to regard the left side as giving a "rotated function," which we evaluate at point $\vec{x}$. The right side tells us that this is the same as the original function evaluated at the point $R^{-1}\vec{x}$, where R^{-1} is the inverse of the rotation matrix corresponding to rotation R. Since $\{Y_{lm}\}$ forms an invariant subspace, we must have:

$$\hat{P}_R Y_{lm} = \sum_{m'=-1}^{l} Y_{lm'} D^l(R)_{m'm}$$

The expansion coefficients, $D^l(R)_{m'm}$, can be regarded as the elements of a matrix $D^l(R)$. As discussed in the note, D^ℓ corresponds to an irreducible representation of the rotation group.

Thus, for a free atom, we have that the degenerate eigenfunctions of a given energy must transform according to an irreducible representation of this group. If the eigenfunctions transform according to the l^{th} representation, the degeneracy of the energy level is $(2l+1)$ (assuming no additional, "accidental" degeneracy).

I remind you of the following:

Definition: Two elements a and b of a group are said to belong to the same "class" (or "equivalence class" or "conjugate class") if there exists a group element g such that $g^{-1}ag = b$.

The first two parts of this problem, are really already done in the note, but here is an opportunity to think about it for yourself:

(a) Show that all proper rotations through the same angle φ, about any axis, belong to the same class of the rotation group.

(b) We will need the character table of this group. Since all elements in the same class have the same character, we pick a convenient element in each class by considering rotations about the z-axis, $R = (\alpha, z)$ (means rotate by angle α about the z-axis). Thus:

$$\hat{P}_{(\alpha,z)} Y_{\ell m} = e^{-im\alpha} Y_{\ell m}$$

(which you should convince yourself of).

Find the character "table" of the rotation group, that is, find $\chi^\ell(\alpha)$, the character of representation D^ℓ for the class of rotations through angle α. If you find an expression for the character in the form of a sum, do the sum, expressing your answer in as simple a form as you can.

(c) At last we are ready to put our atom into a potential with cubic symmetry. Now the symmetry of the free Hamiltonian is broken, and we are left with the discrete symmetry of the cube. The symmetry group of proper rotations of the cube is a group of order 24 with 5 classes. Call this group "O".

Construct the character table for O.

(d) Consider in particular how the f-level ($l = 3$) of the free atom may split when it is placed in the "cubic potential". The seven eigenfunctions which transform according to the irreducible representation D^3 of the full group will most likely not transform according to an irreducible representation of O. On the other hand, since the operations of O are certainly operations of D, the eigenfunctions will generate some representation of O.

Determine the coefficients in the decomposition.

$$D^3 = a_1 O^1 \oplus a_2 O^2 \oplus a_3 O^3 \oplus a_4 O^4 \oplus a_5 O^5,$$

where O^i are the irreducible representations of O. Hence, show how the degeneracy of the 7-fold level may be reduced by the cubic potential. Give the degeneracies of the final levels.

Note that we cannot say anything here about the magnitude of any splittings (which could "accidentally" turn out to be zero!), or even about the ordering of the resulting levels – that depends on the details of the potential, not just its symmetry.

15. We perform an experiment in which we shine a beam of unpolarized white light at a gas of excited hydrogen atoms. We label atomic states by $|n\ell m\rangle$, where ℓ is the total (orbital, we are neglecting spin in this problem) angular momentum, m is the z-component of angular momentum ($L_z|n\ell m\rangle = m|n\ell m\rangle$), and n is a quantum number determining the radial wave function. The light beam is shone along the x-axis.

We are interested in transition rates between atomic states, induced by the light. Since we are dealing with visible light, its wavelength is much larger than the size of the atom. Thus, it is a good first approximation to consider only the interaction of the atomic dipole moment with the electric field of the light beam. That is, the spatial variation in the plane wave e^{ikx}, describing the light beam, may be replaced by the lowest-order term in its expansion, *i.e.*, by 1. Thus, we need only consider the interaction of the dipole moment with the electric field of the light beam, taken to be uniform. The electric dipole moment of the atom is proportional to $e\mathbf{x}$, where $\mathbf{x}$ is the position of the electron relative to the nucleus. Hence, in the "dipole approximation", we are interested in matrix elements of $\mathbf{x}\cdot\mathbf{E}$, where $\mathbf{E}$ is the electric field vector of the light beam.

Calculate the following ratios of transition rates in the dipole approximation:

a)
$$\frac{\Gamma(|23,1,1\rangle \to |1,0,0\rangle)}{\Gamma(|23,1,0\rangle \to |1,0,0\rangle)}$$

b).
$$\frac{\Gamma(|3,1,0\rangle \to |4,2,1\rangle)}{\Gamma(|3,1,-1\rangle \to |4,2,0\rangle)}$$

[Hint: this is an application of the Wigner-Eckart theorem.]

16. It is possible to arrive at the Clebsch-Gordan coefficients for a given situation by "elementary" means, *i.e.*, by considering the action of the raising and lowering operators and demanding orthonormality. Hence, construct a table of Clebsch-Gordan coefficients, using this approach, for a system combining $j_1 = 2$ and $j_2 = 1$ angular momenta. I find it convenient to use the simple notation $|jm\rangle$ for total quantum numbers and $|j_1m_1\rangle|j_2m_2\rangle$ for the individual angular momentum states being added, but you may use whatever notation you find convenient.] You will find (I hope) that you have the freedom to pick certain signs. You are asked to be consistent with the usual conventions where

$$\langle 33|\,(|22\rangle|11\rangle) \geq 0 \tag{527}$$
$$\langle 22|\,(|22\rangle|10\rangle) \geq 0 \tag{528}$$
$$\langle 11|\,(|22\rangle|1\,{-1}\rangle) \geq 0 \tag{529}$$

(in notation $\langle jm|\,(|j_1m_1\rangle|j_2m_2\rangle)$).

17. In our discussion of the Wigner-Eckart theorem, we obtained the reduced matrix element for the angular momentum operator: $\langle j'k'\|J\|j''k''\rangle$. This required knowing the Clebsch-Gordan coefficient $C(1, j, j; 0, m, m)$. By using the general prescription for calculating the $3j$ symbols we developed, calculate the $3j$ symbol

$$\begin{pmatrix} 1 & j & j \\ 0 & m & -m \end{pmatrix},$$

and hence obtain $C(1, j, j; 0, m, m)$.

18. Rotational Invariance and angular distributions: A spin-1 particle is polarized such that its spin direction is along the $+z$ axis. It decays, with total decay rate Γ, to $\pi^+\pi^-$. What is the angular distribution, $d\Gamma/d\Omega$, of the π^+? Note that the $\pi^\pm$ is spin zero. What is the angular distribution if the intial spin projection is zero along the z-axis? Minus one? What is the result if the intial particle is unpolarized (equal probabilities for all spin orientations)?

19. Here is another example of how we can use the rotation matrices to compute the angular distribution in a decay process. Let's try another similar example. Consider a spin one particle, polarized with its angular momentum along the $\pm z$-axis, with equal probabilities. Suppose it decays to two spin-1/2 particles, *e.g.*, an electron and a positron.

    (a) Such the decay occurs with no orbital angular momentum. What is the angular distribution of the decay products, in the frame of the decaying particle?

    (b) If this is an electromagnetic decay to e^+e^-, and the mass of the decaying particle is much larger than the electron mass, the situation is altered, according to relativistic QED. In this case, the final state spins will be oriented in such a way as to give either $m = 1$ or $m = -1$ along the decay axis, where m is the total projected angular momentum. What is the angular distribution of the decay products in this case?