

Physics 195
Course Notes
Angular Momentum
Solutions to Problems
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1 Exercises

1. Prove the theorem we state in this note:

Theorem: The most general mapping $\mathbf{x} \rightarrow \mathbf{x}'$ of R^3 into itself, such that the origin is mapped into the origin, and such that all distances are preserved, is a linear, real orthogonal transformation Q :

$$\mathbf{x}' = Q\mathbf{x}, \quad \text{where } Q^T Q = I, \quad \text{and } Q^* = Q. \quad (1)$$

Hence,

$$\mathbf{x}' \cdot \mathbf{y}' = \mathbf{x} \cdot \mathbf{y} \quad \forall \text{ points } \mathbf{x}, \mathbf{y} \in R^3. \quad (2)$$

For such a mapping, either:

- (a) $\det(Q) = 1$, Q is called a **proper** orthogonal transformation, and is in fact a rotation. In this case,

$$\mathbf{x}' \times \mathbf{y}' = (\mathbf{x} \times \mathbf{y})' \quad \forall \text{ points } \mathbf{x}, \mathbf{y} \in R^3. \quad (3)$$

or,

- (b) $\det(Q) = -1$, Q is called an **improper** orthogonal transformation, and is the product of a reflection (parity) and a rotation. In this case,

$$\mathbf{x}' \times \mathbf{y}' = -(\mathbf{x} \times \mathbf{y})' \quad \forall \text{ points } \mathbf{x}, \mathbf{y} \in R^3. \quad (4)$$

The set of all orthogonal transformations forms a group (denoted $O(3)$), and the set of all proper orthogonal transformations forms a subgroup ($O^+(3)$ or $SO(3)$ of $O(3)$), identical with the set of all rotations.

[You may wish to make use of the following intuitive lemma: Let $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be any three mutually perpendicular unit vectors such that:

$$\mathbf{e}'_3 = \mathbf{e}'_1 \times \mathbf{e}'_2 \quad (\text{right-handed system}). \quad (5)$$

Then there exists a unique rotation $R_{\mathbf{u}}(\theta)$ such that

$$\mathbf{e}'_i = R_{\mathbf{u}}(\theta)\mathbf{e}_i, \quad i = 1, 2, 3. \quad (6)$$

]

Solution: Let us first show that the scalar product is preserved, under a mapping which takes the origin to the origin and which preserves distances. Consider the scalar product between any pair of vectors $\mathbf{x} \in R^3$ and $\mathbf{y} \in R^3$. Under the mapping, we have:

$$\begin{aligned} \mathbf{x}' \cdot \mathbf{y}' &= -\frac{1}{2} [(\mathbf{x}' - \mathbf{y}') \cdot (\mathbf{x}' - \mathbf{y}') - \mathbf{x}' \cdot \mathbf{x}' - \mathbf{y}' \cdot \mathbf{y}'] \\ &= -\frac{1}{2} [d^2(\mathbf{x}', \mathbf{y}') - d^2(\mathbf{x}', 0) - d^2(\mathbf{y}', 0)] \\ &= -\frac{1}{2} [d^2(\mathbf{x}, \mathbf{y}) - d^2(\mathbf{x}, 0) - d^2(\mathbf{y}, 0)] \\ &= \mathbf{x} \cdot \mathbf{y}. \end{aligned} \quad (7)$$

Now we'll show linearity. Let a be any real number. Let $\mathbf{z} = a\mathbf{x} + \mathbf{y}$. We have

$$\begin{aligned} \mathbf{z} \cdot \mathbf{z} &= Q\mathbf{z} \cdot Q\mathbf{z} \\ &= a^2\mathbf{x}^2 + 2a\mathbf{x} \cdot \mathbf{y} + \mathbf{y}^2 \\ &= a^2(Q\mathbf{x})^2 + 2a(Q\mathbf{x}) \cdot (Q\mathbf{y}) + (Q\mathbf{y})^2. \end{aligned} \quad (8)$$

Since a , \mathbf{x} and \mathbf{y} are arbitrary, we must have

$$Q\mathbf{z} = aQ\mathbf{x} + Q\mathbf{y}, \quad (9)$$

that is, Q is a linear operator (this may also be demonstrated by considering a basis). Hence, Q is a 3×3 matrix. Since the mapping is from R^3 into itself, every element of the matrix must be real: To see this explicitly, consider a vector \mathbf{x} with components $x_j = \delta_{jk}$. Then

$$x'_i = \sum_{j=1}^3 Q_{ij}x_j = Q_{ik}. \quad (10)$$

We can pick out each element of Q in this way, and hence each must be real.

Let us look at orthogonality. We must have

$$\begin{aligned}
 \mathbf{x}' \cdot \mathbf{x}' &= \mathbf{x}'^T \mathbf{x}' \\
 &= (Q\mathbf{x})^T Q\mathbf{x} \\
 &= \mathbf{x}^T Q^T Q\mathbf{x} \\
 &= \mathbf{x}^T \mathbf{x}.
 \end{aligned} \tag{11}$$

Since \mathbf{x} is arbitrary, we must have

$$Q^T Q = I, \tag{12}$$

that is, Q is an orthogonal matrix.

The determinant of an orthogonal matrix is either $+1$ or -1 :

$$1 = \det(I) = \det(Q^T Q) = \det(Q^T) \det(Q) = \det(Q)^2. \tag{13}$$

Consider the triple product:

$$(\mathbf{x}' \times \mathbf{y}') \cdot \mathbf{z}' = (Q\mathbf{x} \times Q\mathbf{y}) \cdot Q\mathbf{z} \tag{14}$$

But a triple product may be computed as the determinant of the matrix formed by putting each vector in a column. The transformed triple product thus corresponds to the determinant of the matrix obtained by taking the product of Q times the original triple product matrix. Therefore,

$$(\mathbf{x}' \times \mathbf{y}') \cdot \mathbf{z}' = \det(Q)(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}. \tag{15}$$

For the case $\det(Q) = +1$: Pick an orthonormal basis $\{\mathbf{e}_i\}, i = 1, 2, 3$. The handedness of this basis is preserved under Q , since the triple product is preserved. Let $\mathbf{e}'_i = Q\mathbf{e}_i, i = 1, 2, 3$. The three vectors $\{\mathbf{e}'\}$ constitutes a new orthonormal basis with the same handedness. Since there is a unique rotation relating a basis and a rotated basis (with handedness preserved), there exists a rotation R such that $\mathbf{e}'_i = R\mathbf{e}_i, i = 1, 2, 3$. We see that $R = Q$.

For the case $\det(Q) = -1$ we note that any such matrix may be written as the product of $-I$ and a real orthogonal matrix with $\det = +1$. The remainder of the proof is straightforward.

2. We stated the following generalization of the addition law for tangents:

Theorem: If $R_{\mathbf{e}}(\theta) = R_{\mathbf{e}''}(\theta'')R_{\mathbf{e}'}(\theta')$, and defining:

$$\boldsymbol{\tau} = \mathbf{e} \tan \theta/2 \quad (16)$$

$$\boldsymbol{\tau}' = \mathbf{e}' \tan \theta'/2 \quad (17)$$

$$\boldsymbol{\tau}'' = \mathbf{e}'' \tan \theta''/2, \quad (18)$$

then:

$$\boldsymbol{\tau} = \frac{\boldsymbol{\tau}' + \boldsymbol{\tau}'' + \boldsymbol{\tau}'' \times \boldsymbol{\tau}'}{1 - \boldsymbol{\tau}' \cdot \boldsymbol{\tau}''}. \quad (19)$$

A simple way to prove this theorem is to use SU(2) to represent the rotations, *i.e.*, the rotation $R_{\mathbf{e}}(\theta)$ is represented by the SU(2) matrix $\exp\left(-\frac{i}{2}\theta\mathbf{e} \cdot \boldsymbol{\sigma}\right)$. You are asked to carry out this proof.

Solution: We have

$$R_{\mathbf{e}}(\theta) = R_{\mathbf{e}''}(\theta'')R_{\mathbf{e}'}(\theta'). \quad (20)$$

Represent our rotations with the form:

$$R_{\mathbf{e}}(\theta) = \exp\left(-\frac{i}{2}\theta\mathbf{e} \cdot \boldsymbol{\sigma}\right) = \cos \frac{\theta}{2} - i\mathbf{e} \cdot \boldsymbol{\sigma} \sin \frac{\theta}{2}. \quad (21)$$

Thus,

$$\begin{aligned} \cos \frac{\theta}{2} - i\mathbf{e} \cdot \boldsymbol{\sigma} \sin \frac{\theta}{2} &= \quad (22) \\ &\left(\cos \frac{\theta''}{2} - i\mathbf{e}'' \cdot \boldsymbol{\sigma} \sin \frac{\theta''}{2}\right) \left(\cos \frac{\theta'}{2} - i\mathbf{e}' \cdot \boldsymbol{\sigma} \sin \frac{\theta'}{2}\right) \\ &= \cos \frac{\theta''}{2} \cos \frac{\theta'}{2} - \mathbf{e}'' \cdot \boldsymbol{\sigma} \mathbf{e}' \cdot \boldsymbol{\sigma} \sin \frac{\theta''}{2} \sin \frac{\theta'}{2} \\ &\quad - i \left(\mathbf{e}' \cdot \boldsymbol{\sigma} \cos \frac{\theta''}{2} \sin \frac{\theta'}{2} + \mathbf{e}'' \cdot \boldsymbol{\sigma} \cos \frac{\theta'}{2} \sin \frac{\theta''}{2} \right) \\ &= \cos \frac{\theta''}{2} \cos \frac{\theta'}{2} - \mathbf{e}'' \cdot \mathbf{e}' \sin \frac{\theta'}{2} \sin \frac{\theta''}{2} \\ &\quad - i \left[(\mathbf{e}'' \times \mathbf{e}') \cdot \boldsymbol{\sigma} \sin \frac{\theta'}{2} \sin \frac{\theta''}{2} + \mathbf{e}' \cdot \boldsymbol{\sigma} \cos \frac{\theta''}{2} \sin \frac{\theta'}{2} + \mathbf{e}'' \cdot \boldsymbol{\sigma} \cos \frac{\theta'}{2} \sin \frac{\theta''}{2} \right], \quad (23) \end{aligned}$$

where we have used the identity

$$\mathbf{e}'' \cdot \boldsymbol{\sigma} \mathbf{e}' \cdot \boldsymbol{\sigma} = \mathbf{e}'' \cdot \mathbf{e}' + i(\mathbf{e}'' \times \mathbf{e}') \cdot \boldsymbol{\sigma}. \quad (24)$$

The above result may be separated into real and imaginary parts:

$$\begin{aligned}\cos \frac{\theta}{2} &= \cos \frac{\theta''}{2} \cos \frac{\theta'}{2} - \mathbf{e}'' \cdot \mathbf{e}' \sin \frac{\theta'}{2} \sin \frac{\theta''}{2} & (25) \\ \sin \frac{\theta}{2} &= (\mathbf{e}'' \times \mathbf{e}') \sin \frac{\theta'}{2} \sin \frac{\theta''}{2} + \mathbf{e}' \cos \frac{\theta''}{2} \sin \frac{\theta'}{2} + \mathbf{e}'' \cos \frac{\theta'}{2} \sin \frac{\theta''}{2} & (26)\end{aligned}$$

Dividing these two equations gives us a result which may be expressed in terms of tangents, and thence in the form of the theorem.

3. We made the assertion that if we had an element $u \in SU(2)$ which commuted with every element of the vector space of traceless 2×2 Hermitian matrices, then u must be a multiple of the identity (*i.e.*, either $u = I$ or $u = -I$). Let us demonstrate this, learning a little more group theory along the way.

First, we note that if we have a matrix group, it is possible to generate another matrix representation of the group by replacing each element with another according to the mapping:

$$u \rightarrow v \tag{27}$$

where

$$v = SuS^{-1}. \tag{28}$$

and S is any chosen non-singular matrix.

- (a) Show that if $\{u\}$ is a matrix group, then $\{v|v = SuS^{-1}; S \text{ a non-singular matrix}\}$ is a representation of the group (*i.e.*, the mapping is 1 : 1 and the multiplication table is preserved under the mapping). The representations $\{u\}$ and $\{v\}$ are considered to be equivalent.

A group of unitary matrices is said to be **reducible** if there exists a mapping of the above form such that every element may simultaneously be written in block-diagonal form:

$$M(g) = \begin{pmatrix} A(g) & 0 & 0 \\ 0 & B(g) & 0 \\ 0 & 0 & \ddots \end{pmatrix}$$

$\forall g \in \text{the group}$ ($A(g)$ and $B(g)$ are sub-matrices).

Solution: The mapping is clearly invertible, hence 1:1. It remains to show that the multiplication table is preserved. Suppose $u_3 = u_1u_2$. Under the mapping $u \rightarrow v$:

$$v_3 = Su_3S^{-1} \tag{29}$$

$$= Su_1u_2S^{-1} \tag{30}$$

$$= Su_1S^{-1}Su_2S^{-1} \tag{31}$$

$$= v_1v_2. \tag{32}$$

$$\tag{33}$$

The multiplication table is preserved.

- (b) Show that $SU(2)$ is not reducible (*i.e.*, $SU(2)$ is irreducible).

Solution: There are various ways of demonstrating this, including a “brute-force” calculation. Let us make a simple argument based on counting degrees-of-freedom. We know that it takes three real parameters to uniquely specify an element of $SU(2)$. “Block diagonal” form for $SU(2)$ means diagonal, *i.e.*, if $SU(2)$ is reducible, then every element of the group may be represented simultaneously by a diagonal matrix. A general diagonal 2×2 complex matrix may be parameterized by four real numbers. However, the determinant is preserved under the similarity transformation. Requiring unimodularity reduces the number of degrees of freedom to only two. This is not sufficient, hence $SU(2)$ is not reducible.

- (c) Now prove the following useful lemma: A matrix which commutes with every element of an irreducible matrix group is a multiple of the identity matrix. [Hint: Let B be such a matrix commuting with every element, and consider the eigenvector equation $B\mathbf{x} = \lambda\mathbf{x}$. Then consider the vector $u\mathbf{x}$ where u is any element of the group, and \mathbf{x} is the eigenvector corresponding to eigenvalue λ .]

Solution: Let V be the vector space (“carrier space”) operated on by the elements of the group, G . Consider $B\mathbf{x} = \lambda\mathbf{x}$. We find

$$Bu\mathbf{x} = uB\mathbf{x} = \lambda u\mathbf{x}. \tag{34}$$

But we assume that our group is irreducible. This means that there is no invariant subspace of V under the actions of the group. The set $\{u\mathbf{x} : u \in G\}$ thus spans V . Hence, every element of V is an eigenvector of B , with eigenvalue λ , and therefore $B = \lambda I$.

- (d) Finally, prove the assertion we stated at the beginning of this problem.

Solution: As in the notes, let V_3 be the vector space of traceless 2×2 traceless Hermitian matrices. Suppose $u \in SU(2)$ commutes with every element of V_3 . Any element X of V_3 may be written in the form:

$$X = \sum_{i=1}^3 x_i \sigma_i. \quad (35)$$

We know further that any element g of $SU(2)$ may be expressed as

$$g = e^{-iX} = \exp\left(-i \sum_{i=1}^3 x_i \sigma_i\right) \quad (36)$$

$$= I \cos |\mathbf{x}| - i \left(\frac{\mathbf{x}}{|\mathbf{x}|} \cdot \boldsymbol{\sigma} \right) \sin |\mathbf{x}|. \quad (37)$$

Certainly $[u, I] = 0$, and we are given $[u, X] = 0$, hence $[u, \sigma_i] = 0$, $i = 1, 2, 3$. Thus $[u, g] = 0$, *i.e.*, u commutes with every element of $SU(2)$. By part (c), u is therefore a multiple of the identity.

4. We have discussed rotations using the language of group theory. Let us look at a simple application of group theory in determining “selection rules” implied by symmetry under the elements of the group (where the group is a group of operations, such as rotations). The point is that we can often predict much about the physics of a situation simply by “symmetry” arguments, without resorting to a detailed solution.

Consider a positronium “atom”, *i.e.*, the bound state of an electron and a positron. The relevant binding interaction here is electromagnetism. The electromagnetic interaction doesn’t depend on the orientation of the system, that is, it is invariant with respect to rotations. It also is invariant with respect to reflections. You may wish to convince yourself of these statements by writing down an explicit Hamiltonian, and verifying the invariance.

Thus, the Hamiltonian for positronium is invariant with respect to the group $O(3)$, and hence commutes with any element of this group. Hence, angular momentum and parity are conserved, and the eigenstates of energy are also eigenstates of parity and total angular momentum (J). In fact, the spin and orbital angular momentum degrees

of freedom are sufficiently decoupled that the total spin (S) and orbital angular momentum (L) are also good quantum numbers for the energy eigenstates to an excellent approximation. The ground state of positronium (“parapositronium”) is the 1S_0 state in $^{2S+1}L_J$ spectroscopic notation, where $L = S$ means zero orbital angular momentum. Note that the letter “ S ” in the spectroscopic notation is not the same as the “ S ” referring to the total spin quantum number. Sorry about the confusion, but it’s established tradition...

In the ground state, the positron and electron have no relative orbital angular momentum, and their spins are anti-parallel. The first excited state (“orthopositronium”) is the 3S_1 state, in which the spins of the positron and electron are now aligned parallel with each other. The $^3S_1 - ^1S_0$ splitting is very small, and is analogous to the “hyperfine” splitting in normal atoms.

Positronium decays when the electron and positron annihilate to produce photons. The decay process is also electromagnetic, hence also governed by a Hamiltonian which is invariant under $O(3)$. As a consequence of this symmetry, angular momentum and parity are conserved in the decay.

- (a) We said that parity was a good quantum number for positronium states. To say just what the parity is, we need to anticipate a result from the Dirac equation (sorry): The intrinsic parities of the electron and positron are opposite. What is the parity of parapositronium? Of orthopositronium?

Solution: Both states are $L = 0$, so the spatial wave function depends only on electron-positron separation, and has no angular dependence. Thus, the the spatial wave function is even under parity, and the overall parity is odd due to the opposite intrinsic parities of the electron and positron. $P = -1$ for both states.

- (b) We wish to know whether positronium can decay into two photons. Let us check parity conservation. What are the possible parities of a state of two photons, in the center-of-mass frame? Can you exclude the decay of positronium to two photons on the basis of parity conservation?

Solution: The intrinsic parity of the photon, which happens to be $\eta_\gamma = -1$, doesn’t matter here, since we have two photons,

and $\eta_\gamma^2 = +1$, independent of whether it is ± 1 . Since the spin polarization is unaffected by a spatial reflection, it is only the spatial wave function we must consider. For S -wave ($L = 0$) the spatial wave function does not depend on orientation, and hence $P = +1$. For P -wave ($L = 1$), however, the spatial wave function is odd under parity (*i.e.*, the spatial wave function has a $Y_{1m}(\theta, \phi)$ angular dependence, and $PY_{1m}(\theta, \phi) = Y_{1m}(\pi - \theta, \phi + \pi) = -Y_{1m}(\theta, \phi)$). So, either parity is possible for a state of two photons (assuming we can put them in both $L = 0$ and $L = 1$ states of angular momentum!), and the decay of positronium may not be excluded on this basis.

- (c) Let us consider now whether rotational invariance, *i.e.*, conservation of angular momentum, puts any constraints on the permitted decays of positronium. Can the orthopositronium state decay to two photons? What about the parapositronium state?

Solution: Align a coordinate system so that the z -axis is along the decay direction, in the center-of-mass frame. Since photons carry a spin of ± 1 along their direction of motion, the total spin along the z -axis must be either $S_z = 0$ or $S_z = 2$. The orbital angular momentum must be zero along the direction of motion, $L_z = 0$. Thus, the total angular momentum projection along z is either $J_z = 0$ or $J_z = 2$. By conservation of angular momentum, we cannot have a system with angular momentum 0 or 1 decay into a state with $J_z = 2$, so we exclude this possibility henceforth. If $J_z = 0$, then the two photons must have spin projections which are opposite along z : $S_{1z} = -S_{2z}$. Consider a rotation, $R_y(\pi)$ about the y -axis by angle π . This simply interchanges the two photons, which results in a state indistinguishable from the state before the rotation.

Now consider the effect of $R_y(\pi)$ on the positronium state. On the $J = 0$ state, the rotation has no effect, since the state is invariant with respect to orientation. However, the $J = 1$ state must change sign under this rotation, since it must have $J_z = 0$, and hence must have a wave functions which transforms under rotations according

to $Y_{10}(\theta, \phi)$. That is,

$$R_y(\pi)Y_{10}(\theta, \phi) = R_y(\pi)\sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\cos(\pi - \theta) = -Y_{10}(\theta, \phi). \quad (38)$$

Thus, we find that the parapositronium decay to two photons is permitted, as far as we have checked (it is observed experimentally), but the orthopositronium decay to two photons is forbidden by rotational invariance.

5. The “charge conjugation” operator, C , is an operator that changes all particles into their anti-particles. Consider the group of order 2 generated by the charge conjugation operator. This group has elements $\{I, C\}$, where I is the identity element. The electromagnetic interaction is invariant with respect to the actions of this group. That is, any electromagnetic process for a system of particles should proceed identically if all the particles are replaced by their anti-particles. Hence, C is a conserved quantity. Let’s consider the implications of this for the 1S_0 and 3S_1 positronium decays to two photons. [See the preceding exercise for a discussion of positronium. Note that you needn’t have done that problem in order to do this one.]

- (a) The result of operating C on a photon is to give a photon, *i.e.*, the photon is its own anti-particle, and is thus an eigenstate of C . What is the eigenvalue? That is, what is the “ C -parity” of the photon? You should give your reasoning. No credit will be given for just writing down the answer, even if correct. [Hint: think classically about electromagnetic fields and how they are produced.] Hence, what is the C -parity of a system of n photons?

Solution: Electromagnetic fields, and hence photons, are produced by charged particles. Under C , all electric charges are replaced by their opposites. Thus, under C , all electromagnetic fields are reversed. Thus, the C -parity of the photon is -1 , and a system of n photons will have $C = (-1)^n$.

- (b) It is a bit trickier to figure out the charge conjugation of the positronium states. Since these are states consisting of a particle and its antiparticle, we suspect that they may also be eigenstates

of C . But is the eigenvalue positive or negative? To determine this, we need to know a bit more than we know so far.

Let me give an heuristic argument for the new understanding that we need. First, although we haven't talked about it yet, you probably already know about the "Pauli Exclusion Principle", which states that two identical fermions cannot be in the same state.

Suppose we have a state consisting of two electrons, $|\mathbf{x}_1, \mathbf{S}_1; \mathbf{x}_2, \mathbf{S}_2\rangle$. We may borrow an idea we introduced in our discussion of the harmonic oscillator, and define a "creation operator", $a^\dagger(\mathbf{x}, \mathbf{S})$, which creates an electron at \mathbf{x} with spin \mathbf{S} . Consider the two-electron state:

$$[a^\dagger(\mathbf{x}_1, \mathbf{S}_1)a^\dagger(\mathbf{x}_2, \mathbf{S}_2) + a^\dagger(\mathbf{x}_2, \mathbf{S}_2)a^\dagger(\mathbf{x}_1, \mathbf{S}_1)]|0\rangle, \quad (39)$$

where $|0\rangle$ is the "vacuum" state, with no electrons. But this puts both electrons in the same state, since it is invariant under the interchange $1 \leftrightarrow 2$. Therefore, in order to satisfy the Pauli principle, we must have that

$$a^\dagger(\mathbf{x}_1, \mathbf{S}_1)a^\dagger(\mathbf{x}_2, \mathbf{S}_2) + a^\dagger(\mathbf{x}_2, \mathbf{S}_2)a^\dagger(\mathbf{x}_1, \mathbf{S}_1) = 0 \quad (40)$$

That is, the creation operators anti-commute. To put it another way, if two electrons are interchanged, a minus sign is introduced. You may be concerned that a positron and an electron are non-identical particles, so maybe this has nothing to do with positronium. However, the relativistic description is such that the positron and electron may be regarded as different "components" of the electron (*e.g.*, the positron may be interpreted in terms of "negative-energy" electron states), so this anti-commutation relation is preserved even when creating electrons and positrons.

Determine the C -parity of the 3S_1 and 1S_0 states of positronium, and thus deduce whether decays to two photons are permitted according to conservation of C . [Hint: Consider a positronium state and let C act on it. Relate this back to the original state by appropriate transformations.]

Solution: Our two particle state is of the form:

$$\psi = |e(\mathbf{x}_1, \mathbf{S}_1), \bar{e}(\mathbf{x}_2, \mathbf{S}_2)\rangle \quad (41)$$

Operating with C gives:

$$C\psi = |\bar{e}(\mathbf{x}_1, \mathbf{S}_1), e(\mathbf{x}_2, \mathbf{S}_2)\rangle \quad (42)$$

Now we need to relate this back to the original state to determine the eigenvalue. First, according to the above discussion, let us interchange the electron and positron:

$$C\psi = -|e(\mathbf{x}_2, \mathbf{S}_2), \bar{e}(\mathbf{x}_1, \mathbf{S}_1)\rangle \quad (43)$$

Now, consider the interchange of the spatial coordinates. This has no effect on our S -wave wave functions, but more generally gives a factor of $(-1)^L$:

$$C\psi = -(-1)^L |e(\mathbf{x}_1, \mathbf{S}_2), \bar{e}(\mathbf{x}_2, \mathbf{S}_1)\rangle. \quad (44)$$

Finally, we need to exchange the spins. Our positronium states are labelled with total spin quantum numbers ($\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$). The $S = 1$ state is obtained when the positron and electron spins are aligned with each other. This spin state is symmetric with respect to the interchange of the two spins. In order to obtain an orthogonal spin state then, the $S = 0$ state must be anti-symmetric under spin interchange. We may summarize by saying that the possible spin states have an interchange symmetry given by the factor $-(-1)^S$. Thus,

$$C\psi = (-1)^{L+S} |e(\mathbf{x}_1, \mathbf{S}_1), \bar{e}(\mathbf{x}_2, \mathbf{S}_2)\rangle. \quad (45)$$

The positronium states are thus eigenstates of C with eigenvalue $(-1)^{L+S}$, or, for $L = 0$, $(-1)^S$. The 3S_1 state has $C = -1$, and therefore is forbidden to decay into two photons by C -conservation. The 1S_0 state has $C = +1$, and so the decay into two photons is permitted by C -conservation.

6. Suppose we have a system with total angular momentum 1. We pick a basis corresponding to the three eigenvectors of the z -component of angular momentum, J_z , with eigenvalues $+1, 0, -1$, respectively. We are given an ensemble described by density matrix:

$$\rho = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

- (a) Is ρ a permissible density matrix? Give your reasoning. For the remainder of this problem, assume that it is permissible. Does it describe a pure or mixed state? Give your reasoning.
 - (b) Given the ensemble described by ρ , what is the average value of J_z ?
 - (c) What is the spread (standard deviation) in measured values of J_z ?
7. Let us consider the application of the density matrix formalism to the problem of a spin-1/2 particle (such as an electron) in a static external magnetic field. In general, a particle with spin may carry a magnetic moment, oriented along the spin direction (by symmetry). For spin-1/2, we have that the magnetic moment (operator) is thus of the form:

$$\boldsymbol{\mu} = \frac{1}{2}\gamma\boldsymbol{\sigma}, \quad (46)$$

where $\boldsymbol{\sigma}$ are the Pauli matrices, the $\frac{1}{2}$ is by convention, and γ is a constant, giving the strength of the moment, called the gyromagnetic ratio. The term in the Hamiltonian for such a magnetic moment in an external magnetic field, \mathbf{B} is just:

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}. \quad (47)$$

Our spin-1/2 particle may have some spin-orientation, or “polarization vector”, given by:

$$\mathbf{P} = \langle \boldsymbol{\sigma} \rangle. \quad (48)$$

Drawing from our classical intuition, we might expect that in the external magnetic field the polarization vector will exhibit a precession about the field direction. Let us investigate this.

Recall that the expectation value of an operator may be computed from the density matrix according to:

$$\langle A \rangle = \text{Tr}(\rho A). \quad (49)$$

Furthermore, recall that the time evolution of the density matrix is given by:

$$i\frac{\partial \rho}{\partial t} = [H(t), \rho(t)]. \quad (50)$$

What is the time evolution, $d\mathbf{P}/dt$, of the polarization vector? Express your answer as simply as you can (more credit will be given for right answers that are more physically transparent than for right answers which are not). Note that we make no assumption concerning the purity of the state.

Solution: Let us consider the i th-component of the polarization:

$$i\frac{dP_i}{dt} = i\frac{d\langle\sigma_i\rangle}{dt} \quad (51)$$

$$= i\frac{\partial}{\partial t}\text{Tr}(\rho\sigma_i) \quad (52)$$

$$= i\text{Tr}\left(\frac{\partial\rho}{\partial t}\sigma_i\right) \quad (53)$$

$$= \text{Tr}([H, \rho]\sigma_i) \quad (54)$$

$$= \text{Tr}([\sigma_i, H]\rho) \quad (55)$$

$$= -\frac{1}{2}\gamma\sum_{j=1}^3 B_j\text{Tr}([\sigma_i, \sigma_j]\rho). \quad (56)$$

To proceed further, we need the density matrix for a state with polarization \mathbf{P} . Since ρ is hermitian, it must be of the form:

$$\rho = a(1 + \mathbf{b} \cdot \boldsymbol{\sigma}). \quad (57)$$

But its trace must be one, so $a = 1/2$. Finally, to get the right polarization vector, we must have $\mathbf{b} = \mathbf{P}$.

Thus, we have

$$i\frac{dP_i}{dt} = -\frac{1}{4}\gamma\sum_{j=1}^3 B_j\left\{\text{Tr}[\sigma_i, \sigma_j] + \sum_{k=1}^3 P_k\text{Tr}([\sigma_i, \sigma_j]\sigma_k)\right\}. \quad (58)$$

Now $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, which is traceless. Further, $\text{Tr}([\sigma_i, \sigma_j]\sigma_k) = 4i\epsilon_{ijk}$. This gives the result:

$$\frac{dP_i}{dt} = -\gamma\sum_{j=1}^3\sum_{k=1}^3\epsilon_{ijk}B_jP_k. \quad (59)$$

This may be re-expressed in the vector form:

$$\frac{d\mathbf{P}}{dt} = \gamma\mathbf{P} \times \mathbf{B}. \quad (60)$$

8. Let us consider a system of N spin-1/2 particles (as in the previous problem) per unit volume in thermal equilibrium, in our external magnetic field \mathbf{B} . [Even though we refer to the previous exercise, the solution to this problem does not require solving the previous one.] Recall that the canonical distribution is:

$$\rho = \frac{e^{-H/T}}{Z}, \quad (61)$$

with partition function:

$$Z = \text{Tr} \left(e^{-H/T} \right). \quad (62)$$

Such a system of particles will tend to orient along the magnetic field, resulting in a bulk magnetization (having units of magnetic moment per unit volume), \mathbf{M} .

- (a) Give an expression for this magnetization (don't work too hard to evaluate).

Solution: Let us orient our coordinate system so that the z -axis is along the magnetic field direction. Then $M_x = 0$, $M_y = 0$, and:

$$M_z = N \frac{1}{2} \gamma \langle \sigma_z \rangle \quad (63)$$

$$= N \gamma \frac{1}{2Z} \text{Tr} \left[e^{-H/T} \sigma_z \right], \quad (64)$$

where $H = -\gamma B_z \sigma_z / 2$.

- (b) What is the magnetization in the high-temperature limit, to lowest non-trivial order (this I want you to evaluate as completely as you can!)?

Solution: In the high temperature limit, we'll discard terms of order higher than $1/T$ in the expansion of the exponential: $e^{-H/T} \approx 1 - H/T = 1 + \gamma B_z \sigma_z / 2T$. Thus,

$$M_z = N \gamma \frac{1}{2Z} \text{Tr} \left[(1 + \gamma B_z \sigma_z / 2T) \sigma_z \right] \quad (65)$$

$$= N \gamma^2 B_z \frac{1}{2ZT}. \quad (66)$$

Furthermore,

$$Z = \text{Tr}e^{-H/T} \quad (67)$$

$$= 2 + O(1/T^2). \quad (68)$$

And we have the result:

$$M_z = N\gamma^2 B_z / 4T. \quad (69)$$

This is referred to as the “Curie Law” (for magnetization of a system of spin-1/2 particles).

9. We have discussed Lie algebras (with Lie product given by the commutator) and Lie groups, in our attempt to deal with rotations. At one point, we asserted that the structure (multiplication table) of the Lie group in some neighborhood of the identity was completely determined by the structure (multiplication table) of the Lie algebra. We noted that, however intuitively pleasing this might sound, it was not actually a trivial statement, and that it followed from the “Baker-Campbell-Hausdorff” theorem. Let’s try to tidy this up a bit further here.

First, let’s set up some notation: Let \mathcal{L} be a Lie algebra, and \mathcal{G} be the Lie group generated by this algebra. Let $X, Y \in \mathcal{L}$ be two elements of the algebra. These generate the elements $e^X, e^Y \in \mathcal{G}$ of the Lie group. We assume the notion that if X and Y are close to the zero element of the Lie algebra, then e^X and e^Y will be close to the identity element of the Lie group.

What we want to show is that the group product $e^X e^Y$ may be expressed in the form e^Z , where $Z \in \mathcal{L}$, at least for X and Y not too “large”. Note that the non-trivial aspect of this problem is that, first, X and Y may not commute, and second, objects of the form XY may not be in the Lie algebra. Elements of \mathcal{L} generated by X and Y must be linear combinations of X, Y , and their repeated commutators.

- (a) Suppose X and Y commute. Show explicitly that the product $e^X e^Y$ is of the form e^Z , where Z is an element of \mathcal{L} . (If you think this is trivial, don’t worry, it is!)

Solution: Since X and Y commute, the series expansion for

$$e^Z = e^{X+Y} \quad (70)$$

behaves just like an expansion in ordinary numbers, and hence $e^X e^Y = e^{X+Y}$, and of course $Z = X + Y$ is an element of the algebra.

- (b) Now suppose that X and Y may not commute, but that they are very close to the zero element. Keeping terms to quadratic order in X and Y , show once again that the product $e^X e^Y$ is of the form e^Z , where Z is an element of \mathcal{L} . Give an explicit expression for Z .

Solution: We expand:

$$e^X e^Y = \left[1 + X + \frac{X^2}{2!} + O(3) \right] \left[1 + Y + \frac{Y^2}{2!} + O(3) \right] \quad (71)$$

$$= 1 + X + Y + \frac{X^2}{2!} + \frac{Y^2}{2!} + XY + O(3) \quad (72)$$

and

$$\begin{aligned} e^{X+Y} &= 1 + (X + Y) + \frac{(X + Y)^2}{2!} + O(3) \\ &= 1 + X + Y + \frac{X^2}{2!} + \frac{Y^2}{2!} + XY - \frac{1}{2}[X, Y] + O(3). \end{aligned} \quad (73)$$

Thus,

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+O(3)}, \quad (74)$$

and $Z = X + Y + \frac{1}{2}[X, Y] \in \mathcal{L}$.

- (c) Finally, for more of a challenge, let's do the general theorem: Show that $e^X e^Y$ is of the form e^Z , where Z is an element of \mathcal{L} , as long as X and Y are sufficiently "small". We won't concern ourselves here with how "small" X and Y need to be – you may investigate that at more leisure.

Here are some hints that may help you: First, we remark that the differential equation

$$\frac{df}{du} = Xf(u) + g(u), \quad (75)$$

where $X \in \mathcal{L}$, and letting $f(0) = f_0$, has the solution:

$$f(u) = e^{uX} f_0 + \int_0^u e^{(u-v)X} g(v) dv. \quad (76)$$

This can be readily verified by back-substitution. If g is independent of u , then the integral may be performed, with the result:

$$f(u) = e^{uX} f_0 + h(u, X)g, \quad (77)$$

Where, formally,

$$h(u, X) = \frac{e^{uX} - 1}{X}. \quad (78)$$

Second, if $X, Y \in \mathcal{L}$, then

$$e^X Y e^{-X} = e^{X_c}(Y), \quad (79)$$

where I have introduced the notation “ X_c ” to mean “take the commutator”. That is, $X_c(Y) \equiv [X, Y]$. This fact may be demonstrated by taking the derivative of

$$A(u; Y) \equiv e^{uX} Y e^{-uX} \quad (80)$$

with respect to u , and comparing with our differential equation above to obtain the desired result.

Third, assuming $X = X(u)$ is differentiable, we have

$$e^{X(u)} \frac{d}{du} e^{-X(u)} = -h(1, X(u)_c) \frac{dX}{du}. \quad (81)$$

This fact may be verified by considering the object:

$$B(t, u) \equiv e^{tX(u)} \frac{\partial}{\partial u} e^{-tX(u)}, \quad (82)$$

and differentiating (carefully!) with respect to t , using the above two facts, and finally letting $t = 1$.

One final hint: Consider the quantity

$$Z(u) = \ln(e^{uX} e^Y). \quad (83)$$

The series:

$$\ell(z) = \frac{\ln z}{z-1} = 1 - \frac{z-1}{2} + \frac{(z-1)^2}{3} - \dots \quad (84)$$

plays a role in the explicit form for the result. Again, you are not asked to worry about convergence issues.

Solution: With $Z(u)$ as just defined, consider:

$$e^{Z(u)} \frac{d}{du} e^{-Z(u)} = e^{uX} e^Y \frac{d}{du} (e^{-Y} e^{-uX}) = -X. \quad (85)$$

Thus,

$$X = h(1, Z(u)_c) \frac{dZ}{du}. \quad (86)$$

From our second “fact” above, we may deduce that:

$$e^{Z(u)_c} = e^{uX_c} e^{Y_c}, \quad (87)$$

or, taking the logarithm,

$$Z(u)_c = \ln [e^{uX_c} e^{Y_c}]. \quad (88)$$

We notice that:

$$h(1, \ln X) \ell(X) = 1. \quad (89)$$

Thus,

$$h(1, Z(u)_c) = h[1, \ln (e^{uX_c} e^{Y_c})] = \ell^{-1} (e^{uX_c} e^{Y_c}), \quad (90)$$

and

$$\frac{dZ(u)}{du} = \ell (e^{uX_c} e^{Y_c}) X. \quad (91)$$

Finally, we integrate to obtain:

$$e^X e^Y = \exp \left[Y + \int_0^1 \ell (e^{uX_c} e^{Y_c}) X du \right] \quad (92)$$

We see that the term in the exponential on the right is a linear combination of X , Y , and their repeated commutators, hence is an element of \mathcal{L} . For example, the term of order 3 in the operators is:

$$\frac{1}{12} ([X, [X, Y]] - [Y, [X, Y]]). \quad (93)$$

10. In an earlier exercise we considered the implication of rotational invariance for the decays of positronium states into two photons. Let us generalize and broaden that discussion here. Certain neutral particles (*e.g.*, π^0, η, η') are observed to decay into two photons, and others (*e.g.*,

ω, ϕ, ψ) are not. Let us investigate the selection rules implied by angular momentum and parity conservation (satisfied by electromagnetic and strong interactions) for the decay of a particle (call it “X”) into two photons. Thus, we ask the question, what angular momentum J and parity P states are allowed for two photons?

Set up the problem in the center-of-mass frame of the two photons, with the z -axis in the direction of one photon. We know that since a photon is a massless spin-one particle, it has two possible spin states, which we can describe by its “helicity”, *i.e.*, its spin projection along its direction of motion, which can take on the values ± 1 . Thus, a system of two photons can have the spin states:

$$|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow + \downarrow\uparrow\rangle, |\uparrow\downarrow - \downarrow\uparrow\rangle$$

(The first arrow refers to the photon in the $+z$ direction, the second to the photon in the $-z$ direction, and the direction of the arrow indicates the spin component along the z -axis, NOT to the helicity.) We consider the effect on these states of three operations (which, by parity and angular momentum conservation, should commute with the Hamiltonian):

- P : parity – reverses direction of motion of a particle, but leaves its angular momentum unaltered.
- $R_z(\alpha)$: rotation by angle α about the z -axis. A state with a given value of J_z (z -component of angular momentum) is an eigenstate, with eigenvalue $e^{i\alpha J_z}$.
- $R_x(\pi)$: rotation by π about x -axis. For our two photons, this reverses the direction of motion and also the angular momentum of each photon. For our “X” particle, this operation has the effect corresponding to the effect on the spherical harmonic with the appropriate eigenvalues:

$$R_x(\pi)Y_{JJ_z}(\Omega)$$

(Note that the Y_{lm} functions are sufficient, since a fermion obviously can’t decay into two photons and conserve angular momentum – hence X is a boson, and we needn’t consider $\frac{1}{2}$ -integer spins.)

Make sure that the above statements are intuitively clear to you.

- (a) By considering the actions of these operations on our two-photon states, complete the following table: (one entry is filled in for you)

Photonic Spin State	Transformation		
	P	$R_z(\alpha)$	$R_x(\pi)$
$ \uparrow\uparrow\rangle$	$+\ \uparrow\uparrow\rangle$		
$ \downarrow\downarrow\rangle$			
$ \uparrow\downarrow + \downarrow\uparrow\rangle$			
$ \uparrow\downarrow - \downarrow\uparrow\rangle$			

Solution:

Photonic Spin State	Transformation		
	P	$R_z(\alpha)$	$R_x(\pi)$
$ \uparrow\uparrow\rangle$	$+\ \uparrow\uparrow\rangle$	$e^{2i\alpha} \uparrow\uparrow\rangle$	$ \downarrow\downarrow\rangle$
$ \downarrow\downarrow\rangle$	$ \downarrow\downarrow\rangle$	$e^{2i\alpha} \downarrow\downarrow\rangle$	$ \uparrow\uparrow\rangle$
$ \uparrow\downarrow + \downarrow\uparrow\rangle$	$ \uparrow\downarrow + \downarrow\uparrow\rangle$	$ \uparrow\downarrow + \downarrow\uparrow\rangle$	$ \uparrow\downarrow + \downarrow\uparrow\rangle$
$ \uparrow\downarrow - \downarrow\uparrow\rangle$	$-(\uparrow\downarrow - \downarrow\uparrow\rangle)$	$ \uparrow\downarrow - \downarrow\uparrow\rangle$	$ \uparrow\downarrow - \downarrow\uparrow\rangle$

- (b) Now fill in a table of eigenvalues for a state (*i.e.*, our particle “X”) of arbitrary integer spin J and parity P (or, if states are not eigenvectors, what the transformations yield):

Spin J	Transformation		
	P	$R_z(\alpha)$	$R_x(\pi)$
0	$\begin{Bmatrix} +1 \\ -1 \end{Bmatrix}$		
1	$\begin{Bmatrix} +1 \\ -1 \end{Bmatrix}$		
2, 4, 6, ...	$\begin{Bmatrix} +1 \\ -1 \end{Bmatrix}$		
3, 5, 7, ...	$\begin{Bmatrix} +1 \\ -1 \end{Bmatrix}$		

Note that there may be more than one eigenvalue of $R_z(\alpha)$ for a given row, corresponding to the different possible values of J_z .

Solution: For the rotation by π about the x axis, we know that:

$$Y_{jm}(\theta, \phi) = \sqrt{\frac{2j+1}{4\pi}} D_{m0}^{*j}(\phi, \theta, 0) = \sqrt{\frac{2j+1}{4\pi}} e^{im\phi} d_{m0}^j(\theta). \quad (94)$$

Hence:

$$R_x(\pi)Y_{jm}(\theta, \phi) = Y_{jm}(\pi - \theta, -\phi) \quad (95)$$

$$= \sqrt{\frac{2j+1}{4\pi}} e^{-im\phi} d_{m0}^j(\pi - \theta) \quad (96)$$

$$= \sqrt{\frac{2j+1}{4\pi}} (-)^{j+m} e^{-im\phi} d_{m0}^j(\theta) \quad (97)$$

$$= (-)^{j+m} Y_{jm}^*(\theta, \phi) \quad (98)$$

$$= (-)^{j+m} Y_{j,-m}(\theta, \phi). \quad (99)$$

States with $m = 0$ are eigenstates of $R_x(\pi)$, with eigenvalue $(-)^j$.

Spin J	P	Transformation	
		$R_z(\alpha)$	$R_x(\pi)$
0	$\begin{Bmatrix} +1 \\ -1 \end{Bmatrix}$	1	1
1	$\begin{Bmatrix} +1 \\ -1 \end{Bmatrix}$	$e^{im\alpha}, m = -1, 0, 1$	$Y_{jm} \rightarrow (-)^{m+1} Y_{j,-m}$
2, 4, 6, ...	$\begin{Bmatrix} +1 \\ -1 \end{Bmatrix}$	$e^{im\alpha}, m = -j, \dots, j$	$Y_{jm} \rightarrow (-)^m Y_{j,-m}$
3, 5, 7, ...	$\begin{Bmatrix} +1 \\ -1 \end{Bmatrix}$	$e^{im\alpha}, m = -j, \dots, j$	$Y_{jm} \rightarrow (-)^{m+1} Y_{j,-m}$

- (c) Finally, by using your answers to parts (a) and (b), determine the allowed and forbidden J^P states decaying into two photons, and the appropriate photonic helicity states for the allowed transitions. Put your answer in the form of a table:

Parity	Spin			
	0	1	2,4,...	3,5,...
+1				
-1	$ \uparrow\downarrow - \downarrow\uparrow\rangle$			

Solution: Note that there is no component of orbital angular momentum along the z axis for the two photon states – the entire J_z is due to the spins of the photons. For spin 0, we must have $J_z = 0$ for the two photons. The two $J_z = 0$ states have different parities, hence we make the assignments in the table below.

For spin 1, we must also have $J_z = 0$ for the two photons. But for spin one, the $m = 0$ state is odd under $R_x(\pi)$, while the possible

two photon states are both even. Thus, a spin 1 particle cannot decay to two photons and preserve rotational invariance.

For spin $2, 4, 6, \dots$, we must have $m = 0, \pm 2$ in order to match the possible photon states (property under $R_z(\alpha)$). By considering $R_x(\pi)$ we see that we thus take $m \rightarrow -m$ with no sign change. All of the photon states are consistent with this. It only remains to match the parities of the initial and final states; the result is in the table below.

For spin $3, 5, 7, \dots$, we again must have $m = 0, \pm 2$ in order to match the possible photon states (property under $R_z(\alpha)$). Now, however, $R_x(\pi)$ takes $m \rightarrow -m$ with a sign change. This excludes the two $J_z = 0$ two photon states. We are unable to construct a negative parity two-photon state with $J_z = \pm 2$. We may construct a positive parity state with the desired sign change under $R_x(\pi)$, as shown in the table.

Parity	Spin			
	0	1	2,4,...	3,5,...
+1	$ \uparrow\downarrow + \downarrow\uparrow\rangle$	forbidden	$ \uparrow\uparrow\rangle, \downarrow\downarrow\rangle,$ $ \uparrow\downarrow + \downarrow\uparrow\rangle$	$ \uparrow\uparrow\rangle - \downarrow\downarrow\rangle$
-1	$ \uparrow\downarrow - \downarrow\uparrow\rangle$	forbidden	$ \uparrow\downarrow - \downarrow\uparrow\rangle$	forbidden

You have (I hope) just derived something which is often referred to as “Yang’s theorem”. Note: People often get this wrong, so be careful!

11. We said that if we are given an arbitrary representation, $D(u)$, of $SU(2)$, it may be reduced to a direct sum of irreps $D^r(u)$:

$$D(u) = \sum_r \oplus D^r(u). \quad (100)$$

The **multiplicities** m_j (the number of irreducible representations D^r which belong to the equivalence class of irreducible representations characterized by index j) are unique, and they are given by:

$$m_j = \int_{SU(2)} d(u) \chi^j(u^{-1}) \chi(u), \quad (101)$$

where $\chi(u) = \text{Tr}[D(u)]$.

(a) Suppose you are given a representation, with characters:

$$\chi [u_{\mathbf{e}}(\theta)] = 1 + \frac{\sin \frac{3}{2}\theta + 2 \sin \frac{7}{4}\theta \cos \frac{1}{4}\theta}{\sin \frac{1}{2}\theta}. \quad (102)$$

What irreducible representations appear in the reduction of this representation, with what multiplicities?

Solution: We choose to work with the parameterization of the volume element in Eqn. (103), and with the characters of the irreducible representations given in Eqn. (255):

$$\chi^j(u^{-1}(\theta)) = \chi^j [u_{\mathbf{e}_3}^{-1}(\theta)] = \frac{\sin(j + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta}. \quad (103)$$

Then,

$$\begin{aligned} m_j &= \int_{SU(2)} d(u) \chi_j(u^{-1}) \chi(u) \\ &= \frac{1}{4\pi^2} \int_{4\pi} d\Omega_{\mathbf{e}} \int_0^{2\pi} \sin^2 \frac{\theta}{2} d\theta \frac{\sin(j + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta} \left[1 + \frac{\sin \frac{3}{2}\theta + 2 \sin \frac{7}{4}\theta \cos \frac{1}{4}\theta}{\sin \frac{1}{2}\theta} \right] \\ &= \frac{1}{\pi} \int_0^{2\pi} d\theta \sin \left(\frac{2j+1}{2}\theta \right) \left(\sin \frac{1}{2}\theta + \sin \frac{3}{2}\theta + 2 \sin \frac{7}{4}\theta \cos \frac{1}{4}\theta \right). \end{aligned} \quad (104)$$

We have the orthogonality relation, n and m positive integers:

$$\int_0^{2\pi} \sin \frac{n\theta}{2} \sin \frac{m\theta}{2} d\theta = \pi \delta_{nm} \quad (105)$$

Also,

$$2 \sin \frac{7}{4}\theta \cos \frac{1}{4}\theta = \sin 2\theta + \sin 3\frac{\theta}{2}. \quad (106)$$

Then,

$$\frac{1}{\pi} \int_0^{2\pi} d\theta \sin \left(\frac{2j+1}{2}\theta \right) \sin \frac{\theta}{2} = \pi \delta_{(2j+1)1} = \pi \delta_{j0} \quad (107)$$

$$\frac{1}{\pi} \int_0^{2\pi} d\theta \sin \left(\frac{2j+1}{2}\theta \right) \sin \frac{3\theta}{2} = \pi \delta_{j1} \quad (108)$$

$$\frac{1}{\pi} \int_0^{2\pi} d\theta \sin \left(\frac{2j+1}{2}\theta \right) \sin \frac{4\theta}{2} = \pi \delta_{j\frac{3}{2}} \quad (109)$$

$$\frac{1}{\pi} \int_0^{2\pi} d\theta \sin \left(\frac{2j+1}{2}\theta \right) \sin \frac{3\theta}{2} = \pi \delta_{j1} \quad (110)$$

Thus, the irreducible representations which appear are $j = 0$, with multiplicity $m_0 = 1$, $j = 1$, with multiplicity $m_1 = 2$, and $j = 3/2$, with multiplicity $m_{\frac{3}{2}} = 1$.

- (b) Does the representation we are given look like it could correspond to rotations of a physically realizable system? Discuss.

Solution: This representation is a direct sum of irreducible representations with $j = 0, 1, \frac{3}{2}$. Thus, the state space contains states which are fermionic and states which are bosonic. We know of no way we can build a system which can have both integer angular momenta and half-integer angular momenta.

12. In nuclear physics, we have the notion of “charge independence”, or the idea that the nuclear (strong) force does not depend on whether we are dealing with neutrons or protons. Thus, the nuclear force is supposed to be symmetric with respect to unitary transformations on a space with basis vectors $p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Neglecting the overall phase symmetry, we see that the symmetry group is $SU(2)$, just as for rotations. As with angular momentum, we can generate representations of other dimensions, and work with systems of more than one nucleon. By analogy with angular momentum, we say that the neutron and proton form an “isotopic spin” (or “isospin”) doublet, with

$$|n\rangle = |I = \frac{1}{2}, I_3 = -\frac{1}{2}\rangle \quad (111)$$

$$|p\rangle = |I = \frac{1}{2}, I_3 = +\frac{1}{2}\rangle \quad (112)$$

(The symbol “ T ” is also often used for isospin). Everything you know about $SU(2)$ can now be applied in isotopic spin space.

Study the isobar level diagram of the He^6 , Li^6 , Be^6 nuclear level schemes, and discuss in detail the evidence for charge independence of the nuclear force. These graphs are quantitative, so your discussion should also be quantitative. Also, since these are real-life physical systems, you should worry about real-life effects which can modify an idealized vision.

You may find an appropriate level scheme via a google search (you want a level diagram for the nuclear isobars of 6 nucleons), *e.g.*, at:
http://www.tunl.duke.edu/nucldata/figures/06figs/06_is.pdf

For additional reference, you might find it of interest to look up:
 F. Ajzenberg-Selove, “Energy Levels of Light Nuclei, $A = 5-10$,” *Nucl. Phys.* **A490** 1-225 (1988)
 (see also <http://www.tunl.duke.edu/nucldata/fas/88AJ01.shtml>).

Solution: The isobar diagram for $A = 6$ is shown in Fig. 1. Note that ${}^6\text{He}$, ${}^6\text{Li}$, and ${}^6\text{Be}$ all have a total of 6 nucleons, and differ in how many of those are protons or neutrons. Thus, to test for charge independence, we may compare energy levels for states with the same quantum numbers. Thus, for example, the ${}^6\text{He}$ ground state is $J^P = 0^+$; we should compare it with the lowest $J^P = 0^+$ states of ${}^6\text{Li}$ and ${}^6\text{Be}$. Charge independence says that these levels should be degenerate in energy. Likewise, we should be able to compare the lowest $J^P = 2^+$ energy levels (actually, the lowest 2^+ level for ${}^6\text{Li}$ is identified as an isospin singlet, so we take the first excited 2^+ state in this case – note that isospin singlets are allowed for lithium, with equal numbers of neutrons and protons, but not for helium and beryllium, which must have $T \geq 1$). There is evidently some question about the assignment of the $J^P = 2^+$ level for ${}^6\text{He}$, but let’s assume it is correct for now.

Even without understanding what the energies really mean, we may make a first test of charge independence by asking whether the energy gaps between the levels are independent. We summarize the situation in Table 1:

Table 1: Levels of some comaprable states in the $A = 6$ isobar system.

Energy (MeV)	${}^6\text{He}$	${}^6\text{Li}$	${}^6\text{Be}$
$J^P = 0^+$	4.05	3.563	3.09
$J^P = 2^+$	5.85	5.37	4.76
$J^P = 2^-$		21.0	29
$J^P = 4^-$		25	26
$J^P = 3^-$		26.6	30
$2^+ - 0^+$	1.80	1.81	1.67
$2^- - 0^+$		17.4	26
$4^- - 0^+$		21	23
$3^- - 0^+$		23.0	27

Error bars are not explicitly given, and some further research may be necessary to elucidate exactly how reliable these numbers are. For our purposes, we'll assume that the values are reliable up to ± 5 in the least significant quoted digit. Thus, the $2^+ - 0^+$ energy differences for ${}^6\text{He}$ and ${}^6\text{Li}$ appear to be in excellent agreement, but the ${}^6\text{Li} - {}^6\text{Be}$ difference is something like 0.14 ± 0.07 MeV. While small compared with the overall level spacings, this difference may be real, and is worthy of further investigation.

More generally, we note that the helium and beryllium isobars are related by a simple $n \leftrightarrow p$ reflection in isospin space, so we should expect the numbers of states to be the same, and the spacings to be the same, up to neutron-proton mass difference effects and electromagnetic effects. As already noticed, lithium may exist in isospin zero states, and hence more levels may be expected for lithium. These features are qualitatively borne out by the figure, though the detailed spacings are not the same. It is worth remarking that the Pauli exclusion principle is present, though hidden, in our comment on counting states.

To proceed further, we need to understand what the energy scales really mean. The reference states that the energies in the square brackets are computed according to:

$$E_{[Z, N]} = E(Z, N) - E(3, 3), \quad (113)$$

where $E(Z, N)$ is the ‘‘approximate nuclear energy’’ computed as

$$E(Z, N) = M(Z, N) - ZM(H) - NM(n) - E_C. \quad (114)$$

We may compare the [4.05] relative energy for ${}^6\text{He}$ with the measurement of its beta decay. A little further reading in the references above tells us that the Q value (kinetic energy released) in the decay ${}^6\text{He}(\beta^-){}^6\text{Li}$ is 3.507 MeV. To an excellent approximation, this corresponds to a nuclear mass difference of

$$\Delta M = Q + m_e = 4.02 \text{ MeV}. \quad (115)$$

This is pretty close to the estimate of [4.05] MeV in the figure. Hence, we seem to have a consistent understanding: The ${}^6\text{He}$ nucleus, in its ground state, is 4.02 MeV more massive than the ${}^6\text{Li}$ ground state

nucleus, up to uncertainties in the last digit. Thus, for example, the difference in mass between the ${}^6\text{He}$ ground state nucleus and its ${}^6\text{Li}$ isospin partner is $4.02 - 3.56 = 0.46$ MeV. Considering the neutron-proton mass difference (1.29 MeV), and the difference in estimated Coulomb energy (-1.32 MeV), this difference may be compared with a difference of -0.03 MeV expected from charge independence. This is a significant disagreement. However, the Coulomb correction is large on the scale of the difference, and may not be trustworthy at this level.

13. We defined the “little- d ” functions according to:

$$d_{m_1 m_2}^j(\theta) = D_{m_1 m_2}^j(0, \theta, 0) = \langle j, m_1 | e^{-i\theta J_2} | j, m_2 \rangle$$

where the matrix elements $D_{m_1 m_2}^j(\psi, \theta, \phi)$, parameterized by Euler angles ψ, θ, ϕ , are given in the “standard representation” by:

$$D_{m_1 m_2}^j(\psi, \theta, \phi) = \langle j, m_1 | D^j(u) | j, m_2 \rangle = e^{-i(m_1\psi + m_2\phi)} d_{m_1 m_2}^j(\theta)$$

We note that an explicit calculation for these matrix elements is possible via:

$$D_{m_1 m_2}^j(u) = P_{j m_1}(\partial_x, \partial_y) P_{j m_2}(u_{11}x + u_{21}y, u_{12}x + u_{22}y) \quad (116)$$

where

$$P_{jm}(x, y) \equiv \frac{x^{j+m} y^{j-m}}{\sqrt{(j+m)!(j-m)!}}. \quad (117)$$

Prove the following handy formulas for the $d_{m_1 m_2}^j(\theta)$ functions:

a) $d_{m_1 m_2}^{j*}(\theta) = d_{m_1 m_2}^j(\theta)$ (reality of d^j functions)

Solution: Let us determine SU(2) matrix u for Euler angles $(0, \theta, 0)$, so that we may use Eqn. 116. This must describe a rotation about the two-axis by angle θ . Hence,

$$u = \exp\left(-\frac{i}{2}\theta\sigma_2\right) \quad (118)$$

$$= I \cos \frac{\theta}{2} - i\sigma_2 \sin \frac{\theta}{2} \quad (119)$$

$$= \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}. \quad (120)$$

This is a real matrix, and substitution into Eqn. 116 preserves that reality.

$$b) \quad d_{m_1 m_2}^j(-\theta) = d_{m_2 m_1}^j(\theta)$$

Solution:

$$d_{m_1 m_2}^j(-\theta) = d_{m_1 m_2}^{j*}(-\theta) \quad (121)$$

$$= D_{m_1 m_2}^{j*}(0, -\theta, 0) \quad (122)$$

$$= D_{m_2 m_1}^j(0, \theta, 0) \quad (123)$$

$$= d_{m_2 m_1}^j(\theta). \quad (124)$$

$$c) \quad d_{m_1 m_2}^j(\theta) = (-)^{m_1 - m_2} d_{m_2 m_1}^j(\theta)$$

Solution: Let's write out Eqn. 116 for the little- d functions:

$$d_{m_1 m_2}^j(\theta) = \frac{\partial_x^{j+m_1} \partial_y^{j-m_1}}{\sqrt{(j+m_1)!(j-m_1)!}} \frac{\left(x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2}\right)^{j+m_2} \left(-x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2}\right)^{j-m_2}}{\sqrt{(j+m_2)!(j-m_2)!}} \quad (125)$$

Now, starting with the result of part (b),

$$\begin{aligned} d_{m_2 m_1}^j(\theta) &= d_{m_1 m_2}^j(-\theta) \quad (126) \\ &= \frac{\partial_x^{j+m_1} \partial_y^{j-m_1} \left(x \cos \frac{\theta}{2} - y \sin \frac{\theta}{2}\right)^{j+m_2} \left(x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2}\right)^{j-m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \end{aligned}$$

Now let $x \rightarrow -x$:

$$\begin{aligned} d_{m_2 m_1}^j(\theta) &= (-)^{j+m_1} \frac{\partial_x^{j+m_1} \partial_y^{j-m_1} \left(-x \cos \frac{\theta}{2} - y \sin \frac{\theta}{2}\right)^{j+m_2} \left(-x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2}\right)^{j-m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \\ &= (-)^{j+m_1+j+m_2} \frac{\partial_x^{j+m_1} \partial_y^{j-m_1} \left(x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2}\right)^{j+m_2} \left(-x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2}\right)^{j-m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \\ &= (-)^{2j+2m_2+m_1-m_2} \frac{\partial_x^{j+m_1} \partial_y^{j-m_1} \left(x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2}\right)^{j+m_2} \left(-x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2}\right)^{j-m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \end{aligned}$$

But $(-)^{2j+2m_2+m_1-m_2} = (-)^{m_1-m_2}$, since either $2j$ and $2m_2$ are both odd or both even, hence their sum is always even.

$$d) \quad d_{-m_1, -m_2}^j(\theta) = (-)^{m_1-m_2} d_{m_1 m_2}^j(\theta)$$

Solution:

$$d_{-m_1, -m_2}^j(\theta) = \frac{\partial_x^{j-m_1} \partial_y^{j+m_1} \left(x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2} \right)^{j-m_2} \left(-x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2} \right)^{j+m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \quad (127)$$

Now interchange x and y , and replace the resulting x by $-x$:

$$\begin{aligned} d_{-m_1, -m_2}^j(\theta) &= (-)^{j+m_1} \frac{\partial_x^{j+m_1} \partial_y^{j-m_1} \left(y \cos \frac{\theta}{2} - x \sin \frac{\theta}{2} \right)^{j-m_2} \left(-y \sin \frac{\theta}{2} - x \cos \frac{\theta}{2} \right)^{j+m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \\ &= (-)^{j+m_1-j-m_2} \frac{\partial_x^{j+m_1} \partial_y^{j-m_1} \left(y \cos \frac{\theta}{2} - x \sin \frac{\theta}{2} \right)^{j-m_2} \left(y \sin \frac{\theta}{2} + x \cos \frac{\theta}{2} \right)^{j+m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \\ &= (-)^{m_1-m_2} d_{m_1 m_2}^j(\theta) \end{aligned} \quad (128)$$

$$e) \quad d_{m_1 m_2}^j(\pi - \theta) = (-)^{j-m_2} d_{-m_1, m_2}^j(\theta) = (-)^{j+m_1} d_{m_1, -m_2}^j(\theta)$$

Solution: The second equality follows from part (d):

$$(-)^{j-m_2} d_{-m_1, m_2}^j(\theta) = (-)^{j-m_2} (-)^{m_1+m_2} d_{m_1, -m_2}^j(\theta) \quad (129)$$

$$= (-)^{j+m_1} d_{m_1, -m_2}^j(\theta) \quad (130)$$

For the first equality, using $\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \sin \frac{\theta}{2}$ and $\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos \frac{\theta}{2}$:

$$\begin{aligned} d_{m_1 m_2}^j(\pi - \theta) &= \frac{\partial_x^{j+m_1} \partial_y^{j-m_1} \left(x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2} \right)^{j+m_2} \left(-x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2} \right)^{j-m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \\ &= (-)^{j+m_1} \frac{\partial_x^{j+m_1} \partial_y^{j-m_1} \left(-x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2} \right)^{j+m_2} \left(x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2} \right)^{j-m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \\ &= (-)^{j+m_1} d_{m_1, -m_2}^j(\theta). \end{aligned} \quad (131)$$

$$f) \quad d_{m_1 m_2}^j(2\pi + \theta) = (-)^{2j} d_{m_1 m_2}^j(\theta)$$

Solution: We use $\cos\left(\pi + \frac{\theta}{2}\right) = -\cos\frac{\theta}{2}$ and $\sin\left(\pi + \frac{\theta}{2}\right) = -\sin\frac{\theta}{2}$:

$$\begin{aligned} d_{m_1 m_2}^j(2\pi + \theta) &= \frac{\partial_x^{j+m_1} \partial_y^{j-m_1} \left(-x \cos \frac{\theta}{2} - y \sin \frac{\theta}{2}\right)^{j+m_2} \left(x \sin \frac{\theta}{2} - y \cos \frac{\theta}{2}\right)^{j-m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \\ &= (-)^{j+m_2+j-m_2} \frac{\partial_x^{j+m_1} \partial_y^{j-m_1} \left(x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2}\right)^{j+m_2} \left(-x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2}\right)^{j-m_2}}{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}} \\ &= (-)^{2j} d_{m_1 m_2}^j(\theta). \end{aligned} \tag{132}$$

14. We would like to consider the (qualitative) effects on the energy levels of an atom which is moved from freedom to an external potential (a crystal, say) with cubic symmetry. Let us consider a one-electron atom and ignore spin for simplicity. Recall that the wave function for the case of the free atom looks something like $R_{nl}(r)Y_{lm}(\theta, \phi)$, and that all states with the same n and l quantum numbers have the same energy, *i.e.*, are $(2l+1)$ -fold degenerate. The Hamiltonian for a free atom must have the symmetry of the full rotation group, as there are no special directions. Thus, we recall some properties of this group for the present discussion. First, we remark that the set of functions $\{Y_{lm} : m = -l, -l+1, \dots, l-1, l\}$ for a given l forms the basis for a $(2l+1)$ -dimensional subspace which is invariant under the operations of the full rotation group. [A set $\{\psi_i\}$ of vectors is said to span an *invariant subspace* V_s under a given set of operations $\{P_j\}$ if $P_j\psi_i \in V_s \forall i, j$.] Furthermore, this subspace is “irreducible,” that is, it cannot be split into smaller subspaces which are also invariant under the rotation group.

Let us denote the linear transformation operator corresponding to element R of the rotation group by the symbol \hat{P}_R , *i.e.*:

$$\hat{P}_R f(\vec{x}) = f(R^{-1}\vec{x})$$

The way to think about this equation is to regard the left side as giving a “rotated function,” which we evaluate at point \vec{x} . The right side tells

us that this is the same as the original function evaluated at the point $R^{-1}\vec{x}$, where R^{-1} is the inverse of the rotation matrix corresponding to rotation R . Since $\{Y_{lm}\}$ forms an invariant subspace, we must have:

$$\hat{P}_R Y_{lm} = \sum_{m'=-l}^l Y_{lm'} D^l(R)_{m'm}$$

The expansion coefficients, $D^l(R)_{m'm}$, can be regarded as the elements of a matrix $D^l(R)$. As discussed in the note, D^l corresponds to an irreducible representation of the rotation group.

Thus, for a free atom, we have that the degenerate eigenfunctions of a given energy must transform according to an irreducible representation of this group. If the eigenfunctions transform according to the l^{th} representation, the degeneracy of the energy level is $(2l + 1)$ (assuming no additional, “accidental” degeneracy).

I remind you of the following:

Definition: Two elements a and b of a group are said to belong to the same “class” (or “equivalence class” or “conjugate class”) if there exists a group element g such that $g^{-1}ag = b$.

The first two parts of this problem, are really already done in the note, but here is an opportunity to think about it for yourself:

- (a) Show that all proper rotations through the same angle φ , about any axis, belong to the same class of the rotation group.

Solution: Consider two rotations by angle θ about axes \mathbf{e}' and \mathbf{e}'' : $R_{\mathbf{e}'}(\theta)$ and $R_{\mathbf{e}''}(\theta)$. Let R be the rotation which takes axis \mathbf{e}' into \mathbf{e}'' . Such a rotation must exist, according to the theorem you proved in exercise one. That is:

$$\mathbf{e}'' = R\mathbf{e}'. \quad (133)$$

Then,

$$R_{\mathbf{e}''}(\theta) = RR_{\mathbf{e}'}(\theta)R^{-1}, \quad (134)$$

since this sequence has the effect of first rotating the \mathbf{e}'' axis (thinking of it as embedded in the object to be rotated) to be along \mathbf{e}' , then rotating about this axis, and finally putting the \mathbf{e}'' axis back to its original orientation.

- (b) We will need the character table of this group. Since all elements in the same class have the same character, we pick a convenient element in each class by considering rotations about the z -axis, $R = (\alpha, z)$ (means rotate by angle α about the z -axis). Thus:

$$\hat{P}_{(\alpha,z)} Y_{\ell m} = e^{-im\alpha} Y_{\ell m}$$

(which you should convince yourself of).

Find the character “table” of the rotation group, that is, find $\chi^\ell(\alpha)$, the character of representation D^ℓ for the class of rotations through angle α . If you find an expression for the character in the form of a sum, do the sum, expressing your answer in as simple a form as you can.

Solution: Consider $D_{m_1 m_2}^j(0, 0, \alpha) = e^{im_1 \alpha} \delta_{m_1 m_2}$.

$$\chi_j(\theta) = \text{Tr} [D^j(0, 0, \alpha)] \quad (135)$$

$$= \sum_{m=-j}^j e^{im\alpha} \quad (136)$$

$$= e^{-ij\alpha} \sum_{k=0}^{2j} (e^{i\alpha})^k \quad (137)$$

$$= e^{-ij\alpha} \sum_{k=0}^{\infty} (e^{i\alpha})^k [1 - (e^{i\alpha})^{2j+1}] \quad (138)$$

$$= e^{-ij\alpha} \frac{(e^{i\alpha})^{2j+1} - 1}{e^{i\alpha} - 1} \quad (139)$$

$$= \frac{e^{i(j+\frac{1}{2})\alpha} - e^{-i(j+\frac{1}{2})\alpha}}{e^{i\alpha/2} - e^{-i\alpha/2}} \quad (140)$$

$$= \frac{\sin(j + \frac{1}{2})\alpha}{\sin \frac{\alpha}{2}}. \quad (141)$$

- (c) At last we are ready to put our atom into a potential with cubic symmetry. Now the symmetry of the free Hamiltonian is broken, and we are left with the discrete symmetry of the cube. The symmetry group of proper rotations of the cube is a group of order 24 with 5 classes. Call this group “ O ”.

Construct the character table for O .

Solution: Let the cube be centered at the origin and oriented with its faces centered on the x, y, z axes. The classes of O are:

Class	p_k	Elements
C_1	1	Identity
C_2	6	Rotations by $\pm\frac{\pi}{2}$ about x, y, z
C_3	3	Rotations by π about x, y, z
C_4	6	Rotations by π about lines joining centers of opposite edges
C_5	8	Rotations by $\pm\frac{2\pi}{3}$ about the four diagonals

In constructing the character table, we note:

- The number of irreducible representations is equal to the number of classes, $n_r = n_c = 5$.
- We have the following orthogonality relations among rows and among columns:

$$\sum_{k=1}^{n_c} p_k \chi_{j'}(C_k)^* \chi_{j''}(C_k) = h \delta_{j'j''}, \quad (142)$$

$$\sum_{i=1}^{n_c} \chi_i(C_j)^* \chi_i(C_\ell) = \frac{h}{p_j} \delta_{j\ell}. \quad (143)$$

- The character of a class of dimension d and whose elements are of order m is a sum of d m -th roots of unity.
- The sums of the squares of the dimensions of the irreducible representations is the order of the group:

$$\sum_{n=1}^5 d_n^2 = h = 24. \quad (144)$$

For O , the solution is unique, the irreducible representations must have dimensions 1, 1, 2, 3, 3.

- The second column (dimension 1 irreducible representation) may be determined uniquely by orthogonality with the first column.
- The second column may be obtained by using orthogonality with the first two columns, plus the normalization on the column itself, plus noting that the characters are given by sums of roots of unity. Thus, for example, the character $\chi_3(C_5)$ must be the sum of two of $1, e^{\pm 2\pi i/3}$.

- The same type of considerations are sufficient to uniquely fill in the remaining table. In this case, it is not actually necessary to write any explicit rotation matrices, though that can be done.

The character table is therefore:

1	C_1	1	1	2	3	3
6	C_2	1	-1	0	1	-1
3	C_3	1	1	2	-1	-1
6	C_4	1	-1	0	-1	1
8	C_5	1	1	-1	0	0

- (d) Consider in particular how the f -level ($l = 3$) of the free atom may split when it is placed in the “cubic potential”. The seven eigenfunctions which transform according to the irreducible representation D^3 of the full group will most likely not transform according to an irreducible representation of O . On the other hand, since the operations of O are certainly operations of D , the eigenfunctions will generate some representation of O .

Determine the coefficients in the decomposition.

$$D^3 = a_1 O^1 \oplus a_2 O^2 \oplus a_3 O^3 \oplus a_4 O^4 \oplus a_5 O^5,$$

where O^i are the irreducible representations of O . Hence, show how the degeneracy of the 7-fold level may be reduced by the cubic potential. Give the degeneracies of the final levels.

Note that we cannot say anything here about the magnitude of any splittings (which could “accidentally” turn out to be zero!), or even about the ordering of the resulting levels – that depends on the details of the potential, not just its symmetry.

Solution:

Using the result of part (b), letting $j = 3$:

$$\chi(\alpha) = \frac{\sin(3 + \frac{1}{2})\alpha}{\sin \frac{\alpha}{2}}, \quad (145)$$

we have:

$$\chi(C_1) = 7 \quad (146)$$

$$\chi(C_2) = \frac{\sin \frac{7\pi}{4}}{\sin \frac{\pi}{4}} = -1 \quad (147)$$

$$\chi(C_3) = \frac{\sin \frac{7\pi}{2}}{\sin \frac{\pi}{2}} = -1 \quad (148)$$

$$\chi(C_4) = \frac{\sin \frac{7\pi}{2}}{\sin \frac{\pi}{2}} = -1 \quad (149)$$

$$\chi(C_5) = \frac{\sin \frac{14\pi}{6}}{\sin \frac{2\pi}{6}} = 1. \quad (150)$$

Finally, we may determine the coefficients in the expansion:

$$D^3 = a_1 O^1 \oplus a_2 O^2 \oplus a_3 O^3 \oplus a_4 O^4 \oplus a_5 O^5,$$

or,

$$\chi = \sum_{i=1}^5 a_i \chi_i. \quad (151)$$

The coefficients are determined using the orthogonality relations once more:

$$a_j = \frac{1}{24} \sum_{k=1}^5 p_k \chi_j^*(C_k) \chi(C_k). \quad (152)$$

The result is:

$$D^3 = 0O^1 \oplus 1O^2 \oplus 0O^3 \oplus 1O^4 \oplus 1O^5. \quad (153)$$

15. We perform an experiment in which we shine a beam of unpolarized white light at a gas of excited hydrogen atoms. We label atomic states by $|nlm\rangle$, where ℓ is the total (orbital, we are neglecting spin in this problem) angular momentum, m is the z -component of angular momentum ($L_z|nlm\rangle = m|nlm\rangle$), and n is a quantum number determining the radial wave function. The light beam is shone along the x -axis.

We are interested in transition rates between atomic states, induced by the light. Since we are dealing with visible light, its wavelength is much larger than the size of the atom. Thus, it is a good first approximation to consider only the interaction of the atomic dipole moment with the electric field of the light beam. That is, the spatial variation in the plane wave e^{ikx} , describing the light beam, may be replaced by the

lowest-order term in its expansion, *i.e.*, by 1. Thus, we need only consider the interaction of the dipole moment with the electric field of the light beam, taken to be uniform. The electric dipole moment of the atom is proportional to $e\mathbf{x}$, where \mathbf{x} is the position of the electron relative to the nucleus. Hence, in the “dipole approximation”, we are interested in matrix elements of $\mathbf{x}\cdot\mathbf{E}$, where \mathbf{E} is the electric field vector of the light beam.

Calculate the following ratios of transition rates in the dipole approximation:

$$\text{a) } \frac{\Gamma(|23, 1, 1\rangle \rightarrow |1, 0, 0\rangle)}{\Gamma(|23, 1, 0\rangle \rightarrow |1, 0, 0\rangle)}$$

Solution: In the dipole approximation, we are concerned with matrix elements of the position operator \mathbf{x} . This is a vector operator, with commutation relations

$$[J_k, x_\ell] = i\epsilon_{k\ell m}x_m. \quad (154)$$

Thus, x, y, z are the Cartesian components of a spherical tensor of rank 1, with spherical components:

$$X_1 = -\frac{1}{\sqrt{2}}(x + iy), \quad (155)$$

$$X_0 = z, \quad (156)$$

$$X_{-1} = \frac{1}{\sqrt{2}}(x - iy). \quad (157)$$

If the light is unpolarized, and traveling along the x axis, we have an ensemble of photons with electric fields equally likely to be along the y and z axes. Thus, we are interested in the matrix elements of:

$$y = \frac{i}{\sqrt{2}}(X_1 + X_{-1}) \quad (158)$$

$$z = X_0. \quad (159)$$

That is, we want

$$\begin{aligned} & \Gamma(|n''\ell''m''\rangle \rightarrow |n'\ell'm'\rangle) \\ & \propto \frac{1}{2}|\langle n'\ell'm'|X_1 + X_{-1}|n''\ell''m''\rangle|^2 + |\langle n'\ell'm'|X_0|n''\ell''m''\rangle|^2. \end{aligned} \quad (160)$$

From the notes, the Wigner-Eckart theorem states that the matrix elements for a vector operator may be written:

$$\begin{aligned} \langle (j'm')(k') | Q(1, m) | (j''m'')(k'') \rangle & \quad (161) \\ & = \frac{(-)^{1+j'-j''}}{\sqrt{2j'+1}} C(1j''j'; mm''m') \langle j', k' || Q_1 || j'', k'' \rangle. \end{aligned}$$

Thus, the desired decay rates may be expressed as:

$$\begin{aligned} \Gamma(|n''\ell''m''\rangle \rightarrow |n'\ell'm'\rangle) & \quad (162) \\ & = K(n''\ell''n'\ell') \left[\frac{1}{2} |C(1\ell''\ell'; 1m''m') + C(1\ell''\ell'; -1m''m')|^2 + |C(1\ell''\ell'; 0m''m')|^2 \right]. \end{aligned}$$

The factor of K cancels out in the desired ratios for this problem, so the results depend only on the Clebsch-Gordan coefficients.

Finally,

$$\begin{aligned} \frac{\Gamma(|23, 1, 1\rangle \rightarrow |1, 0, 0\rangle)}{\Gamma(|23, 1, 0\rangle \rightarrow |1, 0, 0\rangle)} & = \frac{\frac{1}{2} |C(110; 110) + C(110; -110)|^2 + |C(110; 010)|^2}{\frac{1}{2} |C(110; 100) + C(110; -100)|^2 + |C(110; 000)|^2} \\ & = \frac{\frac{1}{2} |0 + \frac{1}{\sqrt{3}}|^2 + 0}{0 + |-\frac{1}{\sqrt{3}}|^2} & (163) \end{aligned}$$

$$= \frac{1}{2}. \quad (164)$$

b).
$$\frac{\Gamma(|3, 1, 0\rangle \rightarrow |4, 2, 1\rangle)}{\Gamma(|3, 1, -1\rangle \rightarrow |4, 2, 0\rangle)}$$

[Hint: this is an application of the Wigner-Eckart theorem.]

Solution: We proceed as in part (a):

$$\begin{aligned} \frac{\Gamma(|3, 1, 0\rangle \rightarrow |4, 2, 1\rangle)}{\Gamma(|3, 1, -1\rangle \rightarrow |4, 2, 0\rangle)} & = \frac{\frac{1}{2} |C(112; 101) + C(112; -101)|^2 + |C(112; 001)|^2}{\frac{1}{2} |C(112; 1-10) + C(112; -1-10)|^2 + |C(112; 0-10)|^2} \\ & = \frac{\frac{1}{2} |\frac{1}{\sqrt{2}} + 0|^2 + 0}{\frac{1}{2} |\frac{1}{\sqrt{6}} + 0|^2 + 0} & (165) \end{aligned}$$

$$= 3. \quad (166)$$

16. It is possible to arrive at the Clebsch-Gordan coefficients for a given situation by “elementary” means, *i.e.*, by considering the action of the raising and lowering operators and demanding orthonormality. Hence, construct a table of Clebsch-Gordan coefficients, using this approach, for a system combining $j_1 = 2$ and $j_2 = 1$ angular momenta. I find it convenient to use the simple notation $|jm\rangle$ for total quantum numbers and $|j_1m_1\rangle|j_2m_2\rangle$ for the individual angular momentum states being added, but you may use whatever notation you find convenient.] You will find (I hope) that you have the freedom to pick certain signs. You are asked to be consistent with the usual conventions where

$$\langle 33 | (|22\rangle|11\rangle) \geq 0 \quad (167)$$

$$\langle 22 | (|22\rangle|10\rangle) \geq 0 \quad (168)$$

$$\langle 11 | (|22\rangle|1-1\rangle) \geq 0 \quad (169)$$

(in notation $\langle jm | (|j_1m_1\rangle|j_2m_2\rangle)$).

bf Solution:

The action of the lowering operator on a state $|jm\rangle$ is:

$$J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}|j, m-1\rangle. \quad (170)$$

The Clebsch-Gordan series for combining $j_1 = 2$ and $j_2 = 1$ is:

$$D^2 \otimes D^1 = D^3 \oplus D^2 \oplus D^1. \quad (171)$$

We start with

$$\langle 33 | (|22\rangle|11\rangle) = 1. \quad (172)$$

The lowering operator on $|33\rangle$ gives

$$J_-|33\rangle = \sqrt{6}|32\rangle. \quad (173)$$

Also,

$$J_- (|22\rangle|11\rangle) = 2 (|21\rangle|11\rangle) + \sqrt{2} (|22\rangle|10\rangle) \quad (174)$$

Thus,

$$|32\rangle = \sqrt{\frac{2}{3}} (|21\rangle|11\rangle) + \frac{1}{\sqrt{3}} (|22\rangle|10\rangle). \quad (175)$$

The $|22\rangle$ state is also a linear combination of $(|21\rangle|11\rangle)$ and $(|22\rangle|10\rangle)$, which we obtain by requiring orthogonality with $|32\rangle$ and using the convention specified in the problem statement:

$$|22\rangle = -\frac{1}{\sqrt{3}}(|21\rangle|11\rangle) + \sqrt{\frac{2}{3}}(|22\rangle|10\rangle). \quad (176)$$

Let's show one more, to make sure the idea is clear:

$$J_-|22\rangle = 2|21\rangle \quad (177)$$

$$J_- \left[-\frac{1}{\sqrt{3}}(|21\rangle|11\rangle) + \sqrt{\frac{2}{3}}(|22\rangle|10\rangle) \right] \quad (178)$$

$$\begin{aligned} &= -\frac{1}{\sqrt{3}} \left[\sqrt{6}(|20\rangle|11\rangle) + \sqrt{2}(|21\rangle|10\rangle) \right] + \sqrt{\frac{2}{3}} \left[2(|21\rangle|10\rangle) + \sqrt{2}(|22\rangle|1-1\rangle) \right] \\ &= -\sqrt{2}(|20\rangle|11\rangle) + \sqrt{\frac{2}{3}}(|21\rangle|10\rangle) + \frac{2}{\sqrt{3}}(|22\rangle|1-1\rangle). \end{aligned} \quad (179)$$

Hence,

$$|21\rangle = -\frac{1}{\sqrt{2}}(|20\rangle|11\rangle) + \frac{1}{\sqrt{6}}(|21\rangle|10\rangle) + \frac{1}{\sqrt{3}}(|22\rangle|1-1\rangle). \quad (180)$$

The entire table is shown below.

$$C(21j; m_1 m_2 m)$$

			j						j						j		
m_1	m_2	m	3	3	2	3	2	1	3	2	1	3	2	1	3	2	3
			3	2	2	1	1	1	0	0	0	-1	-1	-1	-2	-2	-3
2	1		1														
2	0			$\frac{1}{\sqrt{3}}$	$\sqrt{\frac{2}{3}}$												
1	1			$\sqrt{\frac{2}{3}}$	$-\frac{1}{\sqrt{3}}$												
2	-1					$\sqrt{\frac{1}{15}}$	$\frac{1}{\sqrt{3}}$	$\sqrt{\frac{3}{5}}$									
1	0					$\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$									
0	1					$\sqrt{\frac{2}{5}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{10}}$									
1	-1								$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{10}}$						
0	0								$\sqrt{\frac{3}{5}}$	0	$-\sqrt{\frac{2}{5}}$						
-1	1								$\frac{1}{\sqrt{5}}$	$-\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{10}}$						
0	-1											$\sqrt{\frac{2}{5}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{10}}$			
-1	0											$\sqrt{\frac{8}{15}}$	$-\frac{1}{\sqrt{6}}$	$-\sqrt{\frac{3}{10}}$			
-2	1											$\frac{1}{\sqrt{15}}$	$-\frac{1}{\sqrt{3}}$	$\sqrt{\frac{3}{5}}$			
-1	-1														$\sqrt{\frac{2}{3}}$	$\frac{1}{\sqrt{3}}$	
-2	0														$\frac{1}{\sqrt{3}}$	$-\sqrt{\frac{2}{3}}$	
-2	-1																1

17. In our discussion of the Wigner-Eckart theorem, we obtained the reduced matrix element for the angular momentum operator: $\langle j'k' || J || j''k'' \rangle$. This required knowing the Clebsch-Gordan coefficient $C(1, j, j; 0, m, m)$. By using the general prescription for calculating the $3j$ symbols we developed, calculate the $3j$ symbol

$$\begin{pmatrix} 1 & j & j \\ 0 & m & -m \end{pmatrix},$$

and hence obtain $C(1, j, j; 0, m, m)$.

Solution: Our general prescription for computing $3j$ symbols is:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = P_{j_1 m_1}(\partial_{x_1}, \partial_{y_1}) P_{j_2 m_2}(\partial_{x_2}, \partial_{y_2}) P_{j_3 m_3}(\partial_{x_3}, \partial_{y_3}) G(\{k\}; \{x\}, \{y\}), \quad (181)$$

where

$$G(\{k\}; \{x\}, \{y\}) \equiv \frac{(x_1y_2 - x_2y_1)^{2k_3}(x_2y_3 - x_3y_2)^{2k_1}(x_3y_1 - x_1y_3)^{2k_2}}{\sqrt{(2k_3)!(2k_1)!(2k_2)!(j_1 + j_2 + j_3 + 1)!}}, \quad (182)$$

where $2k_1, 2k_2, 2k_3$ are non-negative integers given by:

$$2k_3 = j_1 + j_2 - j_3; \quad 2k_1 = j_2 + j_3 - j_1; \quad 2k_2 = j_3 + j_1 - j_2, \quad (183)$$

and

$$P_{jm}(x, y) \equiv \frac{x^{j+m}y^{j-m}}{\sqrt{(j+m)!(j-m)!}}; \quad P_{00} \equiv 1. \quad (184)$$

We also define $P_{jm} \equiv 0$ if $m \notin \{-j, \dots, j\}$.

The k indices in the present case are:

$$2k_3 = 1 \quad (185)$$

$$2k_1 = 2j - 1 \quad (186)$$

$$2k_2 = 1, \quad (187)$$

and thus,

$$G(\{k\}; \{x\}, \{y\}) = \frac{(x_1y_2 - x_2y_1)(x_2y_3 - x_3y_2)^{2j-1}(x_3y_1 - x_1y_3)}{\sqrt{(2j-1)!(2+2j)!}}. \quad (188)$$

Finally,

$$\begin{aligned} \begin{pmatrix} 1 & j & j \\ 0 & m & -m \end{pmatrix} &= P_{10}(\partial_{x_1}, \partial_{y_1})P_{jm}(\partial_{x_2}, \partial_{y_2})P_{j-m}(\partial_{x_3}, \partial_{y_3})G(\{k\}; \{x\}, \{y\}) \\ &= \partial_{x_1}\partial_{y_1} \frac{\partial_{x_2}^{j+m}\partial_{y_2}^{j-m}}{\sqrt{(j+m)!(j-m)!}} \frac{\partial_{x_3}^{j-m}\partial_{y_3}^{j+m}}{\sqrt{(j+m)!(j-m)!}} \\ &\quad \frac{(x_1y_2 - x_2y_1)(x_2y_3 - x_3y_2)^{2j-1}(x_3y_1 - x_1y_3)}{\sqrt{(2j-1)!(2+2j)!}} \\ &= \frac{\partial_{x_2}^{j+m}\partial_{y_2}^{j-m}\partial_{x_3}^{j-m}\partial_{y_3}^{j+m}}{(j+m)!(j-m)!} \frac{(x_2y_3 + x_3y_2)(x_2y_3 - x_3y_2)^{2j-1}}{\sqrt{(2j-1)!(2+2j)!}} \\ &= \frac{\partial_{x_2}^{j+m}\partial_{y_2}^{j-m}\partial_{x_3}^{j-m}\partial_{y_3}^{j+m}}{(j+m)!(j-m)!\sqrt{(2j-1)!(2+2j)!}} \end{aligned}$$

$$\begin{aligned}
& (x_2y_3 + x_3y_2) \sum_{k=0}^{2j-1} \frac{(2j-1)!}{k!(2j-1-k)!} (x_2y_3)^k (-x_3y_2)^{2j-1-k} \\
&= \frac{\partial_{x_2}^{j+m} \partial_{y_2}^{j-m} \partial_{x_3}^{j-m} \partial_{y_3}^{j+m}}{(j+m)!(j-m)!} \sqrt{\frac{(2j-1)!}{(2+2j)!}} \\
&\quad \sum_{k=0}^{2j-1} \frac{(x_2y_3)^{k+1} (-x_3y_2)^{2j-1-k} - (x_2y_3)^k (-x_3y_2)^{2j-k}}{k!(2j-1-k)!} \\
&= \frac{[(j+m)!(j-m)!]^2}{(j+m)!(j-m)!} \sqrt{\frac{(2j-1)!}{(2+2j)!}} \\
&\quad (-)^{j-m} \left[\frac{1}{(j+m-1)!(j-m)!} - \frac{1}{(j+m)!(j-m-1)!} \right] \\
&= (-)^{j-m} \frac{2m}{\sqrt{(2j+2)(2j+1)2j}}. \tag{189}
\end{aligned}$$

Hence, the desired Clebsch-Gordan coefficients are:

$$C(1jj; 0mm) = (-)^{1-j+m} \sqrt{2j+1} \begin{pmatrix} 1 & j & j \\ 0 & m & -m \end{pmatrix} \tag{190}$$

$$= -\frac{m}{\sqrt{j(j+1)}}. \tag{191}$$

This agrees with the result asserted in the notes.

18. Rotational Invariance and angular distributions: A spin-1 particle is polarized such that its spin direction is along the $+z$ axis. It decays, with total decay rate Γ , to $\pi^+\pi^-$. What is the angular distribution, $d\Gamma/d\Omega$, of the π^+ ? Note that the π^\pm is spin zero.

Solution: We assume that the problem statement refers to the center-of-mass frame. The initial state is $|i\rangle = |j=1, m=1\rangle$, where j is the total angular momentum, and m is its projection on the z -axis. There is clearly no azimuth angle dependence in this problem, so the distribution is uniform in azimuth, ϕ . By angular momentum conservation, the final state must have total angular momentum 1. Since the pions are spinless, this means that the orbital angular momentum must be 1. Furthermore, the z projection must also be 1. Thus, the

angular distribution must be:

$$d\Gamma/d\Omega(\theta, \phi) = \Gamma |Y_{11}(\theta, \phi)|^2 = \frac{3}{8\pi} \Gamma \sin^2 \theta. \quad (192)$$

Notice that this is zero at $\theta = 0$ – if the pions are emitted along the z -axis, the orbital angular momentum cannot be along z , so emission at this angle would violate angular momentum conservation.

Let us also solve this problem using the formula we developed in class:

$$d\Gamma/d\Omega(\theta, \phi) \propto \sum_{\lambda_1, \lambda_2} |A_{\lambda_1 \lambda_2}|^2 |d_{m\delta}^j(\theta)|^2, \quad (193)$$

where $\delta \equiv \lambda_1 - \lambda_2$. In this case, the products are spinless, so $\lambda_1 = \lambda_2 = 0$. Thus, as applied to the present problem:

$$d\Gamma/d\Omega(\theta, \phi) \propto \Gamma |d_{10}^1(\theta)|^2 = \Gamma \frac{1}{2} \sin^2 \theta. \quad (194)$$

Normalizing to get a total decay rate of Γ , we recover the result of Eqn. 192.

19. Here is another example of how we can use the rotation matrices to compute the angular distribution in a decay process. Let's try another similar example. Consider a spin one particle, polarized with its angular momentum along the $\pm z$ -axis, with equal probabilities. Suppose it decays to two spin-1/2 particles, *e.g.*, an electron and a positron.

- (a) Such the decay occurs with no orbital angular momentum. What is the angular distribution of the decay products, in the frame of the decaying particle?

Solution: Actually, we are given the answer: If there is no orbital angular momentum, the spatial portion of the wave function is uniform in angle. Thus,

$$\frac{1}{\Gamma} \frac{d\Gamma}{d\Omega} = \frac{1}{4\pi}. \quad (195)$$

But suppose we didn't notice this, or just want to check that our formalism gives the right answer. We must average over the

two initial polarizations $m = \pm 1$, and sum over all possible final helicities (deciding to work in the helicity formalism here). Thus,

$$\frac{1}{\Gamma} \frac{d\Gamma}{d\Omega}(\theta, \phi) = \frac{1}{2} \sum_{m=-1,1} \sum_{\lambda_1=-\frac{1}{2}}^{\frac{1}{2}} \sum_{\lambda_2=-\frac{1}{2}}^{\frac{1}{2}} |A_{\lambda_1\lambda_2}^{(m)}|^2 |d_{m\delta}^1(\theta)|^2. \quad (196)$$

We may simplify a bit by noticing that the helicity amplitudes for $m = -1$ must be related to the $m = 1$ amplitudes by reversing the z direction, *i.e.*, by letting $\theta \rightarrow \pi - \theta$:

$$\begin{aligned} \frac{1}{\Gamma} \frac{d\Gamma}{d\Omega}(\theta, \phi) &= \frac{1}{2} \sum_{\lambda_1=-\frac{1}{2}}^{\frac{1}{2}} \sum_{\lambda_2=-\frac{1}{2}}^{\frac{1}{2}} |A_{\lambda_1\lambda_2}|^2 \left[|d_{1\delta}^1(\theta)|^2 + |d_{1\delta}^1(\pi - \theta)|^2 \right] \\ &= \frac{1}{2} \left\{ (|A_{++}|^2 + |A_{--}|^2) \left[|d_{10}^1(\theta)|^2 + |d_{10}^1(\pi - \theta)|^2 \right] \right. \\ &\quad \left. + |A_{+-}|^2 \left[|d_{11}^1(\theta)|^2 + |d_{11}^1(\pi - \theta)|^2 \right] \right. \\ &\quad \left. + |A_{-+}|^2 \left[|d_{1-1}^1(\theta)|^2 + |d_{1-1}^1(\pi - \theta)|^2 \right] \right\}. \quad (198) \end{aligned}$$

Now,

$$d_{11}^1(\theta) = d_{1-1}^1(\pi - \theta) = \frac{1}{2}(1 + \cos \theta), \quad (199)$$

$$d_{10}^1(\theta) = d_{10}^1(\pi - \theta) = -\frac{1}{\sqrt{2}} \sin \theta, \quad (200)$$

$$d_{1-1}^1(\theta) = d_{11}^1(\pi - \theta) = \frac{1}{2}(1 - \cos \theta). \quad (201)$$

$$(202)$$

Thus,

$$\begin{aligned} \frac{1}{\Gamma} \frac{d\Gamma}{d\Omega}(\theta, \phi) &= \frac{1}{2} \left\{ (|A_{++}|^2 + |A_{--}|^2) \sin^2 \theta \right. \\ &\quad \left. + \frac{1}{2} (|A_{+-}|^2 + |A_{-+}|^2) \left[(1 + \cos \theta)^2 + (1 - \cos \theta)^2 \right] \right\}, \\ &= \frac{1}{2} \left\{ (|A_{++}|^2 + |A_{--}|^2) \sin^2 \theta + (|A_{+-}|^2 + |A_{-+}|^2) (1 + \cos^2 \theta) \right\}. \quad (203) \end{aligned}$$

In order to be consistent with our assumption of $L = 0$, the helicity amplitudes must be constrained such that

$$|A_{++}|^2 + |A_{--}|^2 = |A_{+-}|^2 + |A_{-+}|^2. \quad (204)$$

- (b) If this is an electromagnetic decay to e^+e^- , and the mass of the decaying particle is much larger than the electron mass, the situation is altered, according to relativistic QED. In this case, the final state spins will be oriented in such a way as to give either $m = 1$ or $m = -1$ along the decay axis, where m is the total projected angular momentum. What is the angular distribution of the decay products in this case?

Solution: We should no longer assume $L = 0$ for this part. We may proceed as in part (a), except omitting the $m = 0$ component along the decay axis:

$$\begin{aligned}\frac{1}{\Gamma} \frac{d\Gamma}{d\Omega}(\theta, \phi) &= \frac{1}{2}(|A_{+-}|^2 + |A_{-+}|^2) [(1 + \cos \theta)^2 + (1 - \cos \theta)^2], \\ &= \frac{3}{16\pi}(1 + \cos^2 \theta).\end{aligned}\quad (205)$$

Let us derive this result a bit more intuitively, without using the general formula. Our initial state is either $|j = 1, m = 1\rangle$ or $|1 - 1\rangle$, in the basis where m is the angular momentum along the z axis. The description of the final state is similar, except that now the quantization axis (call it z') is at an angle θ with respect to the z axis. That is, the final state is either $|j = 1, m = 1'\rangle$ or $|1 - 1'\rangle$, where the prime on the ket is intended to indicate the basis in which the z' axis is the quantization axis. We want to determine:

$$\begin{aligned}\frac{1}{\Gamma} \frac{d\Gamma}{d\Omega}(\theta, \phi) &= |A_{+-}^{(1)}\langle 11'|11\rangle|^2 + |A_{+-}^{(-1)}\langle 11'|1 - 1\rangle|^2 \\ &\quad + |A_{-+}^{(1)}\langle 1 - 1'|11\rangle|^2 + |A_{-+}^{(-1)}\langle 1 - 1'|1 - 1\rangle|^2\end{aligned}\quad (206)$$

To determine the indicated scalar products, we would like to express the primed basis vectors in terms of the unprimed basis. A primed basis vector can be obtained by rotating an unprimed basis vector by angle θ ; since we have azimuthal symmetry, let us take the rotation to be about the y axis:

$$|1m'\rangle = R_y(\theta)|1m\rangle.\quad (207)$$

Hence,

$$\frac{1}{\Gamma} \frac{d\Gamma}{d\Omega}(\theta, \phi) = |A_{+-}^{(1)}\langle 11|R_y(\theta)|11\rangle|^2 + |A_{+-}^{(-1)}\langle 1 - 1|R_y(\theta)|11\rangle|^2\quad (208)$$

$$\begin{aligned}
& + |A_{-+}^{(1)} \langle 11 | R_y(\theta) | 1 - 1 \rangle|^2 + |A_{-+}^{(-1)} \langle 1 - 1 | R_y(\theta) | 1 - 1 \rangle|^2 \\
= & |A_{+-}^{(1)} d_{11}^1(\theta)|^2 + |A_{+-}^{(-1)} d_{-11}^1(\theta)|^2 + |A_{-+}^{(1)} d_{1-1}^1(\theta)|^2 + |A_{-+}^{(-1)} d_{-1-1}^1(\theta)|^2.
\end{aligned}$$

Again, we can use the fact that the amplitude for the initial $m = -1$ decay is related to the amplitude for the $m = +1$ decay by reversing the z axis:

$$\begin{aligned}
\frac{1}{\Gamma} \frac{d\Gamma}{d\Omega}(\theta, \phi) &= |A_{+-}^{(1)} d_{11}^1(\theta)|^2 + |A_{+-}^{(1)} d_{11}^1(\pi - \theta)|^2 + |A_{-+}^{(1)} d_{1-1}^1(\theta)|^2 + |A_{-+}^{(1)} d_{1-1}^1(\pi - \theta)|^2 \\
&= |A_{+-} d_{11}^1(\theta)|^2 + |A_{+-} d_{1-1}^1(\theta)|^2 + |A_{-+} d_{1-1}^1(\theta)|^2 + |A_{-+} d_{11}^1(\theta)|^2 \\
&= (|A_{+-}|^2 + |A_{-+}|^2) [d_{11}^1(\theta)^2 + d_{1-1}^1(\theta)^2] \tag{209}
\end{aligned}$$

$$= \frac{3}{16\pi} (1 + \cos^2 \theta). \tag{210}$$