Physics 125
Course Notes
Identical Particles
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1 Introduction

We briefly summarize the issues of dealing with systems of identical particles in this note.

A classical particle, such as a ping pong ball, may be labelled and followed without affecting its dynamics. A collection of identical quantum mechanical particles, such as electrons, cannot be similarly "labelled" and followed without affecting their dynamics. The key distinction is encapsulated as the "overlap of wave functions".

A system with one electron at point $\mathbf{x}_1$ and spin $\mathbf{s}_1$ and another electron at point $\mathbf{x}_2$ and spin $\mathbf{s}_2$ is completely indistinguishable, and hence arguably the same as, the system with the positions and spins switched. Let us develop this idea, for spinless particles initially.

2 Symmetry Under Interchange

Let

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = |\mathbf{x}_1, \mathbf{x_2}\rangle \tag{1}$$

be the wave function for one particle to be at $\mathbf{x}_1$ and an identical particle to be at $\mathbf{x}_2$. Let P_{12} be an operator which interchanges the positions of the two particles:

$$P_{12}|\mathbf{x}_1, \mathbf{x_2}\rangle = |\mathbf{x}_2, \mathbf{x_1}\rangle. \tag{2}$$

Note that

$$\begin{aligned} P_{12}^2|\mathbf{x}_1, \mathbf{x_2}\rangle &= P_{12}|\mathbf{x}_2, \mathbf{x_1}\rangle & (3)\\ &= |\mathbf{x}_1, \mathbf{x_2}\rangle. & (4) \end{aligned}$$

That is, $P_{12}^2 = I$. The pair $\{P_{12}, I\}$ forms an operator group of order two. It is the simplest non-trivial "permutation" group.

Consider the Hamiltonian H for the two particles:

$$H = -\frac{1}{2m}(\nabla_1^2 + \nabla_2^2) + V(\mathbf{x}_1, t) + V(\mathbf{x}_2, t) + U(|\mathbf{x}_1 - \mathbf{x}_2|, t). \tag{5}$$

Note that I have written this in a form such that the interchange of the two particles leaves the Hamiltonian unchanged. If this were not the case, then the two particles would not be "identical". Thus,

$$[P_{12}, H] = 0, \tag{6}$$

and we can find simultaneous eigenfunctions of P_{12} and H. What are the eigenfunctions of P_{12}? Let $\psi_\lambda(\mathbf{x}_1, \mathbf{x}_2)$ be an eigenstate:

$$\begin{aligned} P_{12}\psi_\lambda(\mathbf{x}_1, \mathbf{x}_2) &= \lambda\psi_\lambda(\mathbf{x}_1, \mathbf{x}_2) = \psi_\lambda(\mathbf{x}_2, \mathbf{x}_1) & (7) \\ P_{12}^2\psi_\lambda(\mathbf{x}_1, \mathbf{x}_2) &= \lambda^2\psi_\lambda(\mathbf{x}_1, \mathbf{x}_2) = \psi_\lambda(\mathbf{x}_1, \mathbf{x}_2). & (8) \end{aligned}$$

Thus, $\lambda^2 = 1$, or $\lambda = \pm 1$.

If $\lambda = +1$, $\psi_\lambda(\mathbf{x}_1, \mathbf{x}_2) = \psi_\lambda(\mathbf{x}_2, \mathbf{x}_1)$; the state is *symmetric*. If $\lambda = -1$, $\psi_\lambda(\mathbf{x}_1, \mathbf{x}_2) = -\psi_\lambda(\mathbf{x}_2, \mathbf{x}_1)$; the state is *antisymmetric*. Note that this implies a degeneracy in the energy levels, since there are two states with the same energy. This is referred to as an "*exchange degeneracy*". However, it is experimentally observed that a pair of identical particles is always in an eigenstate of P_{12}, and that eigenstate depends only on the kind of particle. From our present perspective, this has the status of a fundamental principle. However, in quantum field theory, it is a theorem – see, for example, Feynman's 1986 Dirac Memorial Lecture, "The Reason for Antiparticles", Cambridge University Press, or, for a more mathematical treatment, Streater and Wightman's book, "PCT Spin&Statistics and All That".

To describe this observation, we need to add spin to the discussion. When we add spin, P_{12} has the effect (by definition):

$$P_{12}|\mathbf{x}_1, \mathbf{s}_1; \mathbf{x}_2, \mathbf{s_2}\rangle = |\mathbf{x}_2, \mathbf{s}_2; \mathbf{x_1}, \mathbf{s}_1\rangle. \tag{9}$$

That is, the spin quantum numbers are also interchanged. Particles which are symmetric under this interchange are called "*bosons*". Particles which are antisymmetric under this interchange are called "*fermions*". It is found that particles with integer spin (s=0,1,2,...) are always bosons, and particles with half-integer spin ($\frac{1}{2}, \frac{3}{2}, \ldots$) are always fermions. This is the celebrated *connection between spin and statistics.*

An important consequence for fermions is the *Pauli Exclusion Principle*: Two identical fermions cannot be in the same quantum state. We may "demonstrate" this as follows: Let $\{\phi_k(\mathbf{x}, \mathbf{s}) | k = 1, 2, \ldots\}$ be an orthonormal basis for a single particle state. Then a basis for a two particle identical particle state is formed out of the product space:

$$\{\phi_k(\mathbf{x}_1, \mathbf{s}_1)\} \otimes \{\phi_k(\mathbf{x}_2, \mathbf{s}_2)\} \tag{10}$$

Any two-particle wave function can be written in such a basis as:

$$\psi(\mathbf{x}_1, \mathbf{s}_1, \mathbf{x}_2, \mathbf{s}_2) = \sum_j \sum_k A_{jk}\phi_j(\mathbf{x}_1, \mathbf{s}_1)\phi_k(\mathbf{x}_2, \mathbf{s}_2). \tag{11}$$

If the two particles are in the same quantum state, then the coefficients must be symmetric:

$$A_{jk} = A_{kj}. \tag{12}$$

In that case,

$$\begin{aligned}
-\psi(\mathbf{x}_2, \mathbf{s}_2, \mathbf{x}_1, \mathbf{s}_1) &= \psi(\mathbf{x}_1, \mathbf{s}_1, \mathbf{x}_2, \mathbf{s}_2) && (13)\\
&= \sum_j \sum_k A_{jk}\phi_j(\mathbf{x}_1, \mathbf{s}_1)\phi_k(\mathbf{x}_2, \mathbf{s}_2) && (14)\\
&= \sum_j \sum_k A_{kj}\phi_j(\mathbf{x}_1, \mathbf{s}_1)\phi_k(\mathbf{x}_2, \mathbf{s}_2) && (15)\\
&= \sum_j \sum_k A_{jk}\phi_k(\mathbf{x}_1, \mathbf{s}_1)\phi_j(\mathbf{x}_2, \mathbf{s}_2) && (16)\\
&= \psi(\mathbf{x}_2, \mathbf{s}_2, \mathbf{x}_1, \mathbf{s}_1). && (17)
\end{aligned}$$

Thus, $\psi = -\psi$, showing that the two particles cannot be in the same state.

The Pauli exclusion principle is crucial to atomic physics: once an energy level in an atom is filled, any additional electrons *must* go into a different level.

3 Example: Helium

Consider helium, with two electrons. The wave function of these two electrons must be antisymmetric under interchange. We may use this to construct appropriate ground state quantum numbers. As validated in class discussion, we assume that the spin-dependent forces ("fine structure") may be regarded as perturbations, and look for states with the smallest orbital angular momentum, and lowest radial quantum numbers. Thus, we look for a ground

Symmetric ($S=1$)	Antisymmetric ($S=0$)
$\vert{+}{+}\rangle$ $\frac{1}{\sqrt{2}}(\vert{+}{-}\rangle + \vert{-}{+}\rangle$ $\vert{-}{-}\rangle$	$\frac{1}{\sqrt{2}}(\vert{+}{-}\rangle - \vert{-}{+}\rangle$

state where both electrons have $\ell = 0$ and both are in the lowest radial state (principal quantum numbers $n_1 = n_2 = 1$):

$$\psi(\mathbf{x}_1, \mathbf{x}_2) \sim R(r_1, r_2, |\mathbf{x}_1 - \mathbf{x}_2|), \tag{18}$$

where $R(r_1, r_2, |\mathbf{x}_1 - \mathbf{x}_2|) = R(r_2, r_1, |\mathbf{x}_1 - \mathbf{x}_2|)$. That is, the spatial wave function is symmetric under interchange: $\psi(\mathbf{x}_1, \mathbf{x}_2) = \psi(\mathbf{x}_2, \mathbf{x}_1)$. For this to be possible, the part of the wave function involving the spins must be antisymmetric. Labelling the spin states by z-projections, *i.e.*, for single-particle states:

$$|m = +\frac{1}{2}\rangle \;=\; |+\rangle, \tag{19}$$

$$|m = -\frac{1}{2}\rangle \;=\; |-\rangle, \tag{20}$$

we have states for two electrons shown in the table above.

We conclude that the ground state of helium has $S = 0$, in order that the overall wave function be antisymmetric with respect to interchange of the two electrons. In spectroscopic (L-S) notation, the ground state is a ${}^1S_0$ state. We see that the ground state is, in fact, non-degenerate. There is only one state, instead of the possible four spin arrangements otherwise.

4 Example: Isospin and Extended Pauli Principle

We consider an example in nuclear physics, which will lead us to formulating a convenient "extended Pauli principle". Consider the deuteron, made of a proton and a neutron. We guess that this ground state of a proton and a neutron is S-wave ($\ell = 0$), hence the spatial part of the wave function is symmetric under neutron-proton interchange.

Experimentally, we observe no corresponding nn or pp bound states, which we might expect form the charge independence of the nuclear force. Hence, we suspect that the corresponding nn and pp states are forbidden by the Pauli principle. From this inference, we shall deduce the spin of the deuteron. Consider nn or pp in an $\ell = 0$ state. If $S = 1$, the state is symmetric, hence not allowed. If $S = 0$ the state is antisymmetric, hence allowed. As the deuteron does not have nn or pp analogs, we conclude that the spin of the deuteron is $S = 1$.

With this example in mind, we can generalize our discussion of identical particles and obtain a useful bookkeeping tool. Thus, consider isotopic spin, and regard the proton and neutron as identical particles with different isotopic spin projections: The states of a nucleon, N are

$$|p\rangle = |I = \frac{1}{2}, I_3 = \frac{1}{2}\rangle \tag{21}$$

$$|n\rangle = |I = \frac{1}{2}, I_3 = -\frac{1}{2}\rangle. \tag{22}$$

Since the deuteron does not have nn and pp partners, it is an isotopic spin singlet state ($I(d) = 0$):

$$|d\rangle = \frac{1}{\sqrt{2}}(|pn\rangle - |np\rangle). \tag{23}$$

Regarding the p and n as states of identical fermions, we require that the wave function be antisymmetric under interchange of all quantum numbers (space, spin, and isotopic spin). A state of zero orbital angular momentum and $I = 0$ must thus be even under spin interchange. Again the conclusion is $S(d) = 1$.

Note that we haven't actually introduced any new physical principle here, just a bookkeeping device to arrive at the same result as before. Note that the $I = 1$ combination, which is symmetric in isospin space, must have $s = 0$ (antisymmetric), if ℓ is even. We quote our bookkeeping device in the form of an "*Extended Pauli Principle*":

> A state of two fermions, identical in the sense that they belong to the same isotopic spin multiplet, must be antisymmetric under interchange of all quantum numbers (space, spin, and isospin).

Likewise, we quote the corresponding "*Extended Boson Principle*":

A state of two bosons, identical in the same sense, must be symmetric under interchange of all quantum numbers.

The identical particle symmetries have consequences also in scattering, and also are manifest in the discussion of second quantization.

5 Exercises

1. Let us use the Pauli exclusion principle, and the combination of angular momenta, to find the possible states which may arise when more than one electron in an atom are in the same p-shell. Express your answers for the allowed states in the spectroscopic notation: ${}^{2S+1}L_J$, where S is the total spin of the electrons under consideration, L is the total orbital angular momentum, and J is the total angular momentum of the electrons.

   (a) List the possible states for 2 electrons in the same p-shell.

   (b) List the possible states for 3 electrons in the same p-shell.

   (c) List the possible states for 4 electrons in the same p-shell. Hint: before you embark on something complicated for this part, think a bit!

2. The pion (π) is a boson (with spin zero) with isotopic spin $I = 1$.

   (a) Use our "extended "identical" boson symmetry principle to classify the allowed (I, J) values for a system of two pions. Here, J refers to the relative angular momentum, and I to the total isotopic spin, of the two pion state.

   (b) Look up the experimental situation for particles which do and don't decay into two pions. [See, for example: http://pdg.lbl.gov/2002/mxxx.pdf] Discuss what you find. Try to resolve any puzzles, *e.g.*, do you find particles which "ought" to decay to two pions, but don't? Do some decay to two pions when they "shouldn't"?

3. The magnetic dipole moment of the proton is:

$$\boldsymbol{\mu}_p = g_p \frac{e}{2m_p} \mathbf{s}_p, \tag{24}$$

with a measured magnitude corresponding to a value for the gyromagnetic ratio of $g_p = 2 \times (2.792847337 \pm 0.000000029)$. We haven't studied the Dirac equation yet, but the prediction of the Dirac equation for a point spin-1/2 particle is $g = 2$. We may understand the fact that the proton gyromagnetic ratio is not two as being due to its compositeness: In the simple quark model, the proton is made of three quarks, two "ups" (u), and a "down" (d). The quarks are supposed to be point spin-1/2 particles, hence, their gyromagnetic ratios should be $g_u = g_d = 2$ (up to higher order corrections, as in the case of the electron). Let us see whether we can make sense out of the proton magnetic moment.

The proton magnetic moment should be the sum of the magnetic moments of its constituents, and any moments due to their orbital motion in the proton. The proton is the ground state baryon, so we assume that the three quarks are bound together (by the strong interaction) in a state with no orbital angular momentum. By Fermi statistics, the two identical up quarks must have an overall odd wave function under interchange of all quantum numbers. We must apply this with a bit of care, since we are including "color" as one of these quantum numbers here.

Let us look a little at the property of "color". It is the strong interaction analog of electric charge in the electromagnetic interaction. However, instead of one fundamental dimension in charge, there are three color directions, labelled as "red" (r), "blue" (b), and "green" (g). Unitary transformations in this color space, up to overall phases, are described by elements of the group $SU(3)$, the group of unitary unimodular 3×3 matrices. Just like combining spins, we may combine three colors according to a Clebsch-Gordan series, with the result:

$$3 \times 3 \times 3 = 10 + 8 + 8 + 1. \tag{25}$$

We haven't studied this group, so this decomposition into irreducible representations of the product representation is probably new to you. However, the essential aspect here is that there is a singlet in the decomposition. That is, it is possible to combine three colors in such a way as to get a color-singlet state, *i.e.*, a state with no net color charge. These are the states of physical interest for our observed baryons, according to a postulate of the quark model.

(a) After some thought (perhaps involving raising and lowering operators along different directions in this color space), you could probably convince yourself that the singlet state in the decomposition above must be antisymmetric under the interchange of any two colors. Assuming this is the case, write down the color portion of the proton wave function.

(b) Now that you know the color wave function of the quarks in the proton, write down the spin wave function.

(c) Since the proton is uud and its isospin partner the neutron is ddu, and $m_p \approx m_n$, let us make the simplfying assumption that $m_u = m_d$. Given the measured value of g_p, what does your model give for m_u? Recall that the up quark has electric charge $2/3$, and the down quark has electric charge $-1/3$, in units of the positron charge.

(d) Finally, use your results to predict the gyromagnetic moment of the neutron, and compare with observation.