Physics 125
Course Notes
Identical Particles
Solutions to Problems
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1 Exercises

1. Let us use the Pauli exclusion principle, and the combination of angular momenta, to find the possible states which may arise when more than one electron in an atom are in the same p-shell. Express your answers for the allowed states in the spectroscopic notation: ${}^{2S+1}L_J$, where S is the total spin of the electrons under consideration, L is the total orbital angular momentum, and J is the total angular momentum of the electrons.

   (a) List the possible states for 2 electrons in the same p-shell.
   Solution: The possible total orbital angular momenta are

   $$L = 1 \otimes 1 = 0 \oplus 1 \oplus 2, \tag{1}$$

   and the possible total spin states are

   $$S = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1. \tag{2}$$

   We want the overall wave function to be antisymmetric. The odd L states are antisymmetric, while the $S = 0$ state is antisymmetric. Thus, the possible overall antisymmetric states are:

   $$J = (S = 0) \otimes (L = 0) = 0 \tag{3}$$
   $$J = 0 \otimes 2 = 2 \tag{4}$$
   $$J = 1 \otimes 1 = 0 \oplus 1 \oplus 2. \tag{5}$$

   That is, the possible states are:

   $${}^1S_0, {}^1D_2, {}^3P_0, {}^3P_1, {}^3P_2. \tag{6}$$

   (b) List the possible states for 3 electrons in the same p-shell.
   Solution: We note that we cannot give all three electrons $L_z = +1$, hence there is no F state – we'll only be able to make S, P, D

states. Likewise, we can only make an $S = 3/2$ state in S-wave. All other states will have $S = 1/2$. Proceeding in this manner to count all possible states, we have the possibilities:

$$^4S_{\frac{3}{2}},\ ^2D_{\frac{5}{2}},\ ^2D_{\frac{3}{2}},\ ^2P_{\frac{3}{2}},^2P_{\frac{1}{2}},\ ^2S_{\frac{1}{2}}. \tag{7}$$

(c) List the possible states for 4 electrons in the same p-shell. Hint: before you embark on something complicated for this part, think a bit!

Solution: If all six states were occupied, then we would have only one possible state, with

$$\mathbf{L} = \mathbf{S} = \mathbf{J} = 0. \tag{8}$$

Imagine breaking this state up into the contributions from four electrons and two electrons. Then:

$$\begin{aligned} 0 &= \mathbf{L} = \mathbf{L}(2) + \mathbf{L}(4) && (9)\\ 0 &= \mathbf{S} = \mathbf{S}(2) + \mathbf{S}(4) && (10)\\ 0 &= \mathbf{J} = \mathbf{J}(2) + \mathbf{J}(4) && (11) \end{aligned}$$

Hence, the same angular momentum states are available to the four electrons as the two electrons. That is, the possible states are:

$$^1S_0,^1D_2,^3P_0,^3P_1,^3P_2. \tag{12}$$

2. The pion (π) is a boson (with spin zero) with isotopic spin $I = 1$.

(a) Use our "extended "identical" boson symmetry principle to classify the allowed (I, J) values for a system of two pions. Here, J refers to the relative angular momentum, and I to the total isotopic spin, of the two pion state.

Solution: The pion has spin zero, hence $\mathbf{J} = \mathbf{L}$. Odd J is antisymmetric, even symmetric. The pion has isospin 1. Combining two pions can give states with $I = 0, 1, 2$. Even I states are symmetric, odd antisymmetric. To obtain an overall symmetric state, we must have $I + J$ even. These are the allowed possibilities, *i.e.*, if $I = 0$ or $I = 2$, then J must be even, and if $I = 1$ then J must be odd.

(b) Look up the experimental situation for particles which do and don't decay into two pions. [See, for example: http://pdg.lbl.gov/2002/mxxx.pdf] Discuss what you find. Try to resolve any puzzles, *e.g.*, do you find particles which "ought" to decay to two pions, but don't? Do some decay to two pions when they "shouldn't"?

Solution: The lowest mass non-strange meson above the pion is the η, with $I = 0$ and $J = 0$. According to our rule, it ought to decay to two pions. But it doesn't! The problem is parity, which is conserved in strong and electromagnetic interactions, and which is odd for the η. Two pions in even L have even spatial parity. Thus, we may augment our rule to include that $P = (-)^J$.

The next state is the $f_0(600)$. It also has $I = J = 0$, but $P = +$. It ought to decay to two pions, and it does!

The $\rho(770)$ has $I = J = 1$, and negative parity. It should decay to two pions and it does. Another success!

The $\omega(782)$ has $I = 0$ and $J = 1$. It shouldn't decay to two pions. It does though. However, it dominantly decays to three pions, which is kinematically less likely than two pions. Thus, the two pion decay is at least suppressed. Indeed, it is smaller than the clearly electromagnetic $\pi^0\gamma$ decay. Hence, we conclude that our prediction is still all right, up to the electromagnetic interaction. The electromagnetic interaction clearly violates isospin conservation (*e.g.*, the proton and neutron interact differently electromagnetically).

The K_S^0 meson has $I = 1/2$, so it shouldn't decay to two pions according to our rule. But it does! This decay is occurs via the weak interaction; isospin is not conserved in the weak interaction.

3. The magnetic dipole moment of the proton is:

$$\boldsymbol{\mu}_p = g_p \frac{e}{2m_p} \mathbf{s}_p, \tag{13}$$

with a measured magnitude corresponding to a value for the gyromagnetic ratio of $g_p = 2 \times (2.792847337 \pm 0.000000029)$. We haven't studied the Dirac equation yet, but the prediction of the Dirac equation for a point spin-1/2 particle is $g = 2$. We may understand the fact that the proton gyromagnetic ratio is not two as being due to its compositeness: In the simple quark model, the proton is made of three quarks, two “ups” (u), and a “down” (d). The quarks are supposed to be point spin-1/2 particles, hence, their gyromagnetic ratios should be $g_u = g_d = 2$ (up to higher order corrections, as in the case of the electron). Let us see whether we can make sense out of the proton magnetic moment.

The proton magnetic moment should be the sum of the magnetic moments of its constituents, and any moments due to their orbital motion in the proton. The proton is the ground state baryon, so we assume that the three quarks are bound together (by the strong interaction) in a state with no orbital angular momentum. By Fermi statistics, the two identical up quarks must have an overall odd wave function under interchange of all quantum numbers. We must apply this with a bit of care, since we are including “color” as one of these quantum numbers here.

Let us look a little at the property of “color”. It is the strong interaction analog of electric charge in the electromagnetic interaction. However, instead of one fundamental dimension in charge, there are three color directions, labelled as “red” (r), “blue” (b), and “green” (g). Unitary transformations in this color space, up to overall phases, are described by elements of the group $SU(3)$, the group of unitary unimodular 3×3 matrices. Just like combining spins, we may combine three colors according to a Clebsch-Gordan series, with the result:

$$3 \times 3 \times 3 = 10 + 8 + 8 + 1. \tag{14}$$

We haven’t studied this group, so this decomposition into irreducible representations of the product representation is probably new to you. However, the essential aspect here is that there is a singlet in the decomposition. That is, it is possible to combine three colors in such a way as to get a color-singlet state, *i.e.*, a state with no net color charge. These are the states of physical interest for our observed baryons, according to a postulate of the quark model.

(a) After some thought (perhaps involving raising and lowering operators along different directions in this color space), you could probably convince yourself that the singlet state in the decomposition above must be antisymmetric under the interchange of any two colors. Assuming this is the case, write down the color portion of the proton wave function.

(b) Now that you know the color wave function of the quarks in the proton, write down the spin wave function.

(c) Since the proton is uud and its isospin partner the neutron is ddu, and $m_p \approx m_n$, let us make the simplfying assumption that $m_u = m_d$. Given the measured value of g_p, what does your model give for m_u? Recall that the up quark has electric charge 2/3, and the down quark has electric charge $-1/3$, in units of the positron charge.

(d) Finally, use your results to predict the gyromagnetic moment of the neutron, and compare with observation.

Solution: Note that there are six permutations of the three colors among the three quarks, if no two quarks have the same color. The completely antisymmetric combination of three colors is:

$$\frac{1}{\sqrt{6}}(|rgb\rangle - |rbg\rangle + |brg\rangle - |bgr\rangle + |gbr\rangle - |grb\rangle). \tag{15}$$

To construct the spin wave function, we first note that the three spin-1/2 quarks must combine in such a way as to give an overall spin-1/2 for the proton. Second, since the space wave function is symmetric, and the color wave function is antisymmetric, the spin wave function of the two up quarks must be symmetric. Thus, the two up quarks are in a spin 1 state. To give the spin wave function of the proton, let us pick the z axis to be along the spin direction. Then the spin state is:

$$|\frac{1}{2}\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|11;\frac{1}{2} - \frac{1}{2}\rangle - \frac{1}{\sqrt{3}}|10;\frac{1}{2}\frac{1}{2}\rangle. \tag{16}$$

The magnetic moment of the proton in this model is thus:

$$\mu_p = \frac{2}{3}(2\mu_u - \mu_d) + \frac{1}{3}\mu_d = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d. \tag{17}$$

Hence,

$$g_p \frac{e}{2m_p} = \frac{4}{3} 2 \frac{2}{3} \frac{e}{2m_u} - \frac{1}{3} 2 \left(-\frac{1}{3}\right) \frac{e}{2m_d}. \tag{18}$$

With $g_p = 5.58$, $m_p = 938$ MeV, and $m_u = m_d$, we obtain

$$m_u = 2m_p/g_p = 336 \text{ MeV}. \tag{19}$$

The neutron wave function may be obtained from the proton wave function by interchanging the u and d quark labels. Thus,

$$\mu_n = \frac{2}{3}(2\mu_d - \mu_u) + \frac{1}{3}\mu_u = \frac{4}{3}\mu_d - \frac{1}{3}\mu_u. \tag{20}$$

We predict the gyromagnetic moment of the nuetron to be:

$$\frac{\mu_n}{\mu_p} = \frac{\frac{4}{3}\mu_d - \frac{1}{3}\mu_u}{\frac{4}{3}\mu_u - \frac{1}{3}\mu_d} \tag{21}$$

$$= \frac{\frac{4}{3}\left(-\frac{1}{3}\right) - \frac{1}{3}\frac{2}{3}}{\frac{4}{3}\frac{2}{3} - \frac{1}{3}\left(-\frac{1}{3}\right)} \tag{22}$$

$$= -\frac{2}{3}. \tag{23}$$

That is, we predict (neglecting the mass difference) $g_n = -3.72$. This may be compared with the observed value of -3.83.