Physics 195a
Course Notes
Preliminaries – Solutions to Exercises
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1 Exercises

1. We have reviewed the Bohr atom briefly. This atom is held together by electromagnetism. We may consider the very similar problem of a gravitationally bound "Bohr atom". Part of the point of this problem is to give you some practice making calculations with our simple $\hbar = c = 1$ units, so I hope you will attempt your calculations in this spirit. The gravitational potential between two objects of masses m_1 and m_2 is

$$V = -\frac{Gm_1 m_2}{r},$$

where $r = |\mathbf{r}_1 - \mathbf{r}_2|$, and

$$\begin{aligned} G_N &= 6.67 \times 10^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}, & (1) \\ &= 6.71 \times 10^{-39}\,\mathrm{GeV}^{-2}, & (2) \end{aligned}$$

is Newton's constant.

(a) With the same quantization condition as for Bohr's atom, find the formulas for the energy levels, relative velocities, and relative separations, for a "gravity atom". Let n be the quantum level (*i.e.*, $n = 1, 2, 3, \ldots$ is the orbital angular momentum).

Solution: First, let us write:

$$V = \frac{-GmM}{r},$$

where $M = m_1 + m_2$ and m is the reduced mass. The virial theorem argument gives, for a circular orbit,

$$T = -\frac{1}{2}V = \frac{1}{2}\frac{GmM}{r} = \frac{1}{2}mv^2,$$

where v is the relative speed and non-relativistic motion is assumed (but should be checked!). Now we set $mvr = n$, and see what this implies. The energy is:

$$E = T + V = -\frac{1}{2}\frac{GMm}{r} = -\frac{1}{2}mv^2,$$

and thus

$$\begin{aligned}r &= \frac{n^2}{GMm}\frac{1}{m} &(3)\\ v &= \frac{GMm}{n} &(4)\\ E &= -\frac{1}{2}\left(\frac{GMm}{n}\right)^2 m. &(5)\end{aligned}$$

(b) Find n for the earth-sun system. Is quantum mechanics important here?

Solution: The period is approximately $T = 2\pi r/v = \pi 10^7$ s, and the scale m is approximately $m \approx m_{oplus} = 6 \times 10^{27}$ g. The orbit radius is $r = 1.5 \times 10^{13}$ cm. Thus,

$$rv = \frac{2\pi r^2}{T} = \frac{n}{m}, \tag{6}$$

or

$$\begin{aligned}n &= \frac{2\pi m r^2}{T}\\ &= 2\pi\frac{6\times 10^{27}(\mathrm{g})[0.511(\mathrm{MeV})/(9.11\times 10^{-28}(\mathrm{g})]2.25\times 10^{52}(\mathrm{fm\text{-}fm})}{\pi 10^7(\mathrm{s})3\times 10^{23}(\mathrm{fm/s})200(\mathrm{MeV\text{-}fm})}\\ &= \frac{2\times 6\times .511\times 2.25}{9.11\times 3\times 2}10^{27+28+52-7-23-2}\\ &\approx 2\times 10^{74}. &(7)\end{aligned}$$

The levels for such high values of n are extremely closely spaced, hence the "quantumness" is essentially invisible.

(c) What would r be for the ground state of the earth-sun system (give answer in meters)?

Solution: Since $r \propto n^2$, and $r = 1.5 \times 10^{26}$ fm for $n = 2 \times 10^{74}$, we have for $n = 1$:

$$r_1 = \frac{1.5\times 10^{26}}{4\times 10^{148}}\ \mathrm{fm} \approx 4\times 10^{-123}\ \mathrm{fm}, \tag{8}$$

a very small distance indeed!

(d) Consider a gravitationally bound system of two neutrons, where we suppose that we have "turned off" other potentially important interactions (such as the strong interaction). What is the ground state energy (in GeV) and the ground state separation (in meters) for this system? Is gravity important compared with other forces for a real system of two neutrons in a real ground state?

Solution: We'll approximate the mass of the neutron as $m_n = 1$ GeV for this exercise. The reduced mass is $m = 1/2m_n$ and the total mass is $M = 2m_n$. The ground state energy corresponds to $n = 1$ in this model:

$$E_1 = -\frac{1}{2}G^2(2m)^2\frac{m}{2}^2\frac{m}{2} = -\frac{1}{4}G^2m^5. \tag{9}$$

Newton's constant is, approximately,

$$G = 6.7 \times 10^{-39}\ \text{GeV}^{-2}. \tag{10}$$

Thus,

$$E_1 \approx -10^{-77}\ \text{GeV}. \tag{11}$$

The ground state separation between the two neutrons in this model is

$$r_1 \approx \frac{2}{Gm^3} \approx \frac{2 \times (200\ \text{MeV-fm}) \times (10^{-15}\ \text{m/fm}}{(6.7 \times 10^{-39}\ \text{GeV}) \times (10^3\ \text{MeV/GeV})} \approx 6 \times 10^{22}\ \text{m}. \tag{12}$$

Nuclear sizes are of order 1 fm, and nuclear binding energies are of order MeV (*e.g.*, the binding energy of the deuteron is approximately 2 MeV). It appears that gravity is unimportant compared with the strong interaction (at least) for a system of two neutrons which are not far apart.

(e) Gravity may be important in a quantum mechanical system if the interaction potential is comparable with other forces. For example, the electromagnetic strength is characterized by $e^2 = \alpha = 1/137$, and the strong interaction (though not strictly Coulombic), is characterized by a larger number, $\alpha_s \approx 1$. Supposing we have a system of two equal masses, determine the mass (in GeV), such that the corresponding gravitational interaction strength is equal to one. That is, the potential energy should be given simply by $1/r$ (*cf.*, $V = e^2/r$ for two electrons in electromagnetism). [This mass has a name; do you know what it is?]

Solution: We wish to find the mass M_P such that

$$|V(r)| = \frac{1}{r} = \frac{GM_P^2}{r}. \tag{13}$$

Hence, the "Planck mass" is

$$M_P = \frac{1}{\sqrt{G}} = \frac{1}{\sqrt{6.7 \times 10^{-39}\ \text{GeV}^{-2}}} \approx 10^{19}\ \text{GeV}. \tag{14}$$

2. Resonances I: An ensemble of neutron decays, observed from time $t = 0$, will exhibit the characteristic radioactive decay law:

$$N(t) = N(0)e^{-t/\tau}, \tag{15}$$

where $\tau = 886.7 \pm 1.9\,\text{s}$ (Review of Particle Properties, 2000) is the mean lifetime of the neutron (convince yourself that this should be the case). Also, if an ensemble of neutrons were each weighed very precisely, it would be found that the mass distribution has a (small) width. The full width at half maximum (FWHM) of this distribution is typically denoted by Γ. The mean lifetime is inversely related to this width: $\tau = 1/\Gamma$.

(a) What is the width of the neutron, in electron volts?

Solution: The width of the mass distribution is

$$\Gamma = \frac{1}{\tau} = \frac{(197.3\ \text{Mev-fm})(10^6\ \text{eV/MeV})}{(886.7 \pm 1.9\ \text{s})(3 \times 10^{23}\ \text{fm/s}}) = (7.417{\pm}0.016){\times}10^{-19}\ \text{eV}$$

(b) Classical resonances: Let us probe a bit what this "width" means, and what it has to do with decay rate. We'll start with a classical example: Consider a damped, driven harmonic oscillator:

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = \cos\omega t$$

Determine the frequency response of this classical oscillator, *i.e.*, determine the square of the amplitude of oscillation as a function of ω. (Why the square?)

Solution: The square of the amplitude is interesting because it is proportional to the stored energy. The general solution to the motion is the sum of the solution to the homogeneous equation plus a particular solution to the inhomogeneous equation. Because the solution to the homogeneous equation is transient (due to the damping term), we need only concern ourselves with a particular solution, for which we take:

$$x(t) = A\cos(\omega t + \phi). \tag{16}$$

We substitute this into the differential equation, obtaining:

$$-\omega^2 A\cos(\omega t+\phi) - \Gamma\omega A\sin(\omega t+\phi) + \omega_0^2 A\cos(\omega t+\phi) = \cos\omega t. \tag{17}$$

We may evaluate this at $\omega t + \phi = 0$,

$$-A(\omega^2 - \omega_0^2) = \cos\phi, \tag{18}$$

and at $\omega t + \phi = \pi/2$,

$$-A\Gamma\omega = \sin\phi. \tag{19}$$

Using $\sin^2\phi + \cos^2\phi = 1$, we thus find

$$A^2 = \frac{1}{(\omega^2-\omega_0^2)^2 + \omega^2\Gamma^2}. \tag{20}$$

(c) In the limit of a narrow resonance, what is the full width at half maximum of the distribution you found in part b.

Solution: The peak of the distribution is at

$$\frac{d(1/A^2)}{d\omega^2} = 0 = 2(\omega^2 - \omega_0^2) + \Gamma^2, \tag{21}$$

or $\omega^2 = \omega_0^2 - \Gamma^2/2$. The value of A^2 at the peak is therefore

$$A^2_{\mathrm{max}} = \frac{1}{\Gamma^2(\omega_0^2 - \Gamma^2/4)} \approx \frac{1}{\omega_0^2\Gamma^2}, \tag{22}$$

the approximation being in the "narrow resonance" limit, $\omega_0 \gg \Gamma$. Thus, the amplitude at half-maximum is

$$\frac{1}{2}\frac{1}{\Gamma^2(\omega_0^2 - \Gamma^2/4)} = \frac{1}{(\omega_h^2 - omega_0^2)^2 + \omega_h^2\Gamma^2}, \tag{23}$$

where ω_h are the frequencies at half maximum. Taking the inverse of both sides, we obtain a quadratic equation in $\omega_h^2 - \omega_0^2$, with solution

$$\omega_h^2 - \omega_0^2 = \frac{\Gamma^2}{2}\left[-1 + \sqrt{(2\omega_0/\Gamma)^2 - 1}\right]. \tag{24}$$

In the narrow resonance limit, we find that FWHM$= \Gamma$.

(d) Now suppose the driving force is turned off. How does the energy stored in the oscillator change with time? Find the "lifetime" of this oscillator. In this classical example, I mean the time it takes for the oscillator to reach e^{-1} of its original energy. Your answer should be very simply related to your answer for part c).

Solution: Once the driving force is removed, the oscillations damp down according to the solution to the homogeneous equation:

$$x(t) = Ae^{-\Gamma t/2}\cos\left(\sqrt{\omega_0^2 - \Gamma^2/4}\, t\right). \tag{25}$$

The stored energy is proportional to the square of the maximum amplitude of each cycle, in the limit $\omega_0 \gg \Gamma$. Hence the time to decay to $1/e$ of the initial stored energy is just $1/\Gamma$.

3. The Bohr model for the atom, while wrong, gave some remarkable agreement with experiment, and a means of estimating atomic scales as long as we didn't push the model too hard. Let's play with another, wrong, model, in this problem, the "plum pudding" model of J.J. Thomson. In this model, the atomic electrons are embedded in a region of neutralizing positive charge. We assume that, within the radius of the atom, the positive charge is uniformly distributed. We consider an atom with atomic number Z, but which has been ionized such that only one electron remains. A simple calculation with Maxwell's divergence equation yields that the electric field inside the atom, due to the positive charge distribution, is linear in radius. The force on the electron can therefore be written as:

$$F_r = -eE_r = -\alpha k r,$$

where $-e$ is the electron charge, $\alpha = e^2$, and k depends on the radius R of the atom, and on Z.

(a) Write down the Hamiltonian for the electron in this "atom", making sure you define any quantities not already defined above. Assume that any contribution from radii greater than R may be neglected.

Solution: Let m be the reduced mass of the electron-postive charge system, and $\boldsymbol{p}$ be the relative momentum vector with magnitude $p = mv$. The Hamiltonian is:

$$H = \frac{p^2}{2m} + \frac{1}{2}\alpha k r^2.$$

(b) Assuming circular orbits, use the Bohr quantization condition on angular momentum to derive the allowed energy spectrum.

Solution: The Bohr quantization condition is that the angular momentum is quantized:

$$L = mvr = n,$$

for an orbit of radius r. To get the energy spectrum, we may consider that the centripetal and electrostatic forces must be equal and opposite:

$$\alpha k r = mv^2/r.$$

Hence,

$$\begin{aligned} v^2 &= \frac{\alpha k}{m} r^2 \\ r^2 &= \frac{n}{\sqrt{\alpha k m}} \\ v^2 &= \frac{n}{m}\sqrt{\frac{\alpha k}{m}} \end{aligned}$$

And finally:

$$E_n = n\sqrt{\frac{\alpha k}{m}},$$

where $n = 1, 2, \ldots$

(c) For the hydrogen atom, the ground state energy is -13.6 eV. Note that we have to be a bit careful now in discussing the energy, since we need to know where we our reference (zero) is. Also, our formula for the spectrum in part (b) must have a cut-off somewhere due to the finite atom size. However, we'll circumvent the complication here by considering the difference between the ground state and the first excited state, which for hydrogen is about 10 eV. Using this fact, determine the radius of the ground state orbit, expressing your answer in Å to one significant digit.

[We really ought to check that your answer to part (c) is consistent with the model, *i.e.*, whether the radius obtained is less than R or not. But I'll leave this to your own amusement.]

Solution: The ground state radius, r_0, in tems of this energy difference, $\Delta = \sqrt{\alpha k/m} = 10\,\text{eV}$, is:

$$\begin{aligned} r_0 &= \sqrt{\frac{1}{\Delta m}} \\ &= \sqrt{\frac{1}{(10\,\text{eV})(0.5\,\text{Mev})}(10^6\,\text{eV/MeV})}(200\,\text{MeV fm})(10^{-5}\,\text{Å/fm}) \\ &= 0.9\,\text{Å} \end{aligned}$$

(1 Å is an acceptable answer here also.)