

**Physics 195a**  
**Course Notes**  
**Solving the Schrödinger Equation: Resolvents**  
**Solutions to Exercises**  
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## 1 Exercises

1. Prove identities (12) and (13).
2. Prove the power series expansion for resolvent  $G(z)$  (Eqn. 2):

$$G(z) = G(z_0) \sum_{n=0}^{\infty} [(z - z_0)G(z_0)]^n.$$

You may wish to attempt to do this either “directly”, or via iteration on the identity of Eqn. 11.

3. Prove the result in Eqn. ??.
4. Let’s consider once again the Hamiltonian

$$H = -\frac{1}{2m} \frac{d^2}{dx^2}, \tag{1}$$

but now in configuration space  $x \in [a, b]$  (“infinite square well”).

- (a) Construct the Green’s function,  $G(x, y; z)$  for this problem.

**Please do not look at this solution until you have turned in problem set number 8**

**Solution:** To construct the Green’s function, we look for solutions to:

$$Hu(x) = -\frac{1}{2m} \frac{d^2}{dx^2} u(x; z) = zu(x; z). \tag{2}$$

The solutions may be expressed in the form

$$u(x; z) = A \sin \rho(x + \alpha), \tag{3}$$

where  $\rho \equiv \sqrt{2mz}$ . The left and right solutions, giving the boundary conditions  $u(a) = u(b) = 0$  are thus:

$$u_L(x; z) = A \sin \rho(x - a) \tag{4}$$

$$u_R(x; z) = B \sin \rho(b - x). \tag{5}$$

The Green's function is:

$$G(x, y; z) \equiv \frac{2m}{W(z)} [u_L(x; z)u_R(y; z)\theta(y - x) + u_L(y; z)u_R(x; z)\theta(x - y)], \quad (6)$$

where the Wronskian is

$$W(z) \equiv u'_L(x; z)u_R(x; z) - u_L(x; z)u'_R(x; z). \quad (7)$$

Since the Wronskian is independent of  $x$ , we pick a convenient place to evaluate it,  $x = b$ :

$$W(z) = AB\rho \sin \rho(b - a). \quad (8)$$

Hence, the Green's function is

$$\begin{aligned} G(x, y; z) &= \frac{2m}{\rho \sin \rho(b - a)} \\ &\times \left[ \sin \rho(x - a) \sin \rho(b - y)\theta(y - x) \right. \\ &\quad \left. + \sin \rho(y - a) \sin \rho(b - x)\theta(x - y) \right]. \end{aligned} \quad (9)$$

(b) From your answer to part (a), determine the spectrum of  $H$ .

**Solution:** We first remark that  $G$  is regular at  $\rho = 0$ , hence at  $z = 0$ . The poles appear at:

$$\sqrt{2mz}(b - a) = k\pi, \quad k = \pm 1, \pm 2, \dots \quad (10)$$

That is, the eigenvalues of  $H$  are:

$$\omega_k = \frac{\pi^2 k^2}{2m(b - a)^2}, \quad k = 1, 2, \dots \quad (11)$$

(c) Notice that, using

$$G(x, y; z) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\phi_k^*(y)}{\omega_k - z}, \quad (12)$$

the normalized eigenstate,  $\phi_k(x)$ , can be obtained by evaluating the residue of  $G$  at the pole  $z = \omega_k$ . Do this calculation, and check that your result is properly normalized.

**Please do not look at this solution until you have turned in problem set number 9**

**Solution:** We need to evaluate the residue at the pole  $z = \omega_k$ :

$$-\phi_k(x)\phi_k^*(y) = \lim_{z \rightarrow \omega_k} (z - \omega_k)G(x, y; z) \quad (13)$$

$$\begin{aligned} &= \lim_{z \rightarrow \omega_k} (z - \omega_k) \frac{2m}{\rho \sin \rho(b-a)} \\ &\quad \times \left[ \sin \rho(x-a) \sin \rho(b-y) \theta(y-x) \right. \\ &\quad \left. + \sin \rho(y-a) \sin \rho(b-x) \theta(x-y) \right]. \end{aligned} \quad (14)$$

Hence, consider

$$\begin{aligned} \lim_{z \rightarrow \omega_k} (z - \omega_k) \sqrt{\frac{2m}{z}} \frac{1}{\sin \left[ \sqrt{2mz}(b-a) \right]} &= \frac{(2m)^2(b-a)}{k\pi} \lim_{z \rightarrow \omega_k} \frac{z - \frac{\pi^2 k^2}{2m(b-a)^2}}{\sin \left[ \sqrt{2mz}(b-a) \right]} \\ &= (-1)^k \frac{2}{b-a}. \end{aligned} \quad (15)$$

Thus,

$$\sin \frac{\pi k}{b-a}(b-y) = \sin \frac{\pi k}{b-a}(b-a+a-y) \quad (16)$$

$$\begin{aligned} &= \sin \pi k \cos \pi k \frac{a-y}{b-a} - \sin \pi k \frac{y-a}{b-a} \cos \pi k \\ &= (-1)^{k+1} \sin \pi k \frac{y-1}{b-a}. \end{aligned} \quad (17)$$

$$\begin{aligned} \phi_k(x)\phi_k^*(y) &= (-1)^{k+1} \frac{2}{b-a} \left[ \sin \frac{\pi k}{b-a}(x-a) \sin \frac{\pi k}{b-a}(b-y) \theta(y-x) \right. \\ &\quad \left. + \sin \frac{\pi k}{b-a}(y-a) \sin \frac{\pi k}{b-a}(b-x) \theta(x-y) \right]. \end{aligned} \quad (18)$$

$$\begin{aligned} &= (-1)^{k+1} \frac{2}{b-a} \left[ \sin \frac{\pi k}{b-a}(x-a) \sin \frac{\pi k}{b-a}(b-y) \theta(y-x) \right. \\ &\quad \left. + \sin \frac{\pi k}{b-a}(y-a) \sin \frac{\pi k}{b-a}(b-x) \theta(x-y) \right]. \end{aligned} \quad (19)$$

$$\begin{aligned} &= (-1)^{k+1} \frac{2}{b-a} \left[ \sin \frac{\pi k}{b-a}(x-a) \sin \frac{\pi k}{b-a}(y-a) \theta(y-x) (-1)^{k+1} \right. \\ &\quad \left. + \sin \frac{\pi k}{b-a}(y-a) \sin \frac{\pi k}{b-a}(x-a) \theta(x-y) (-1)^{k+1} \right]. \\ &= \frac{2}{b-a} \sin \frac{\pi k}{b-a}(x-a) \sin \frac{\pi k}{b-a}(y-a). \end{aligned} \quad (20)$$

Finally,

$$\phi_k(x) = \sqrt{\frac{2}{b-a}} \sin \frac{\pi k}{b-a}(x-a). \quad (21)$$

The normalization may be checked by:

$$\begin{aligned}
\int_a^b dx |\phi_k(x)|^2 &= \frac{2}{b-a} \int_a^b dx \left| \sin \frac{\pi k}{b-a} (x-a) \right|^2 \\
&= 2 \int_0^1 dy \sin^2 \pi k y \\
&= 1.
\end{aligned} \tag{22}$$

- (d) Consider the limit  $a \rightarrow -\infty$ ,  $b \rightarrow \infty$ . Show, in this limit that  $G(x, y; z)$  tends to the Green's function we obtain in this note for this Hamiltonian on  $x \in (-\infty, \infty)$ :

$$G(x, y; z) = i \sqrt{\frac{m}{2z}} e^{i\rho|x-y|}. \tag{23}$$

**Solution:** We wish to take the limit  $a \rightarrow -\infty$ ,  $b \rightarrow \infty$  on the Green's function:

$$\begin{aligned}
G(x, y; z) &= \frac{2m}{\rho \sin \rho(b-a)} \\
&\times \left[ \sin \rho(x-a) \sin \rho(b-y) \theta(y-x) \right. \\
&\quad \left. + \sin \rho(y-a) \sin \rho(b-x) \theta(x-y) \right].
\end{aligned} \tag{24}$$

To do this, consider,

$$\sin \rho(x-a) = \frac{1}{2i} \left[ e^{i\rho(x-a)} - e^{-i\rho(x-a)} \right]. \tag{25}$$

Let  $\Im \rho > 0$ . Then:

$$\sin \rho(x-a) \rightarrow -\frac{1}{2i} e^{-i\rho(x-a)}, \quad -a \rightarrow \infty; \tag{26}$$

$$\sin \rho(b-a) \rightarrow -\frac{1}{2i} e^{-i\rho(b-a)}, \quad -a \rightarrow \infty, b \rightarrow \infty; \tag{27}$$

$$\sin \rho(y-a) \rightarrow -\frac{1}{2i} e^{-i\rho(y-a)}, \quad -a \rightarrow \infty; \tag{28}$$

$$\sin \rho(b-x) \rightarrow -\frac{1}{2i} e^{-i\rho(b-x)}, \quad b \rightarrow \infty; \tag{29}$$

$$\sin \rho(b-y) \rightarrow -\frac{1}{2i} e^{-i\rho(b-y)}, \quad b \rightarrow \infty. \tag{30}$$

Hence,

$$G(x, y; z) = \frac{2m e^{-i\rho(b-a)}}{\rho - 2i}$$

$$\begin{aligned}
& \times [\theta(y-x)e^{-i\rho(x-a)}e^{-i\rho(b-y)} + \theta(x-y)e^{-i\rho(b-x)}e^{-i\rho(y-a)}] \\
= & i\sqrt{\frac{m}{2z}} [\theta(y-x)e^{i\rho(y-x)} + \theta(x-y)e^{i\rho(x-y)}] \\
= & i\sqrt{\frac{m}{2z}} e^{i\rho|y-x|}, \tag{31}
\end{aligned}$$

as hoped.