**Physics 125**
**Course Notes**
**Scattering**
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# Contents



# 1 Introduction

Much of what we learn in physical investigations is performed by "scattering" objects off of one another. The quantum mechanical theory for scattering processes is thus an important subject. We have already gotten a glimpse into how to deal with scattering in our discussion of time-dependent perturbation theory, and in our investigation of angular momentum. We carry this development further in this note and derive more of the important concepts and tools for dealing with scattering. Many of the results we obtain will in fact be applicable to relativistic processes.

We start by developing the basic physical picture and mathematical formalism for a simple situation with the essential features. That is, we begin by considering the problem of the scattering of a spinless particle of mass $m > 0$ from a fixed, static, spherically symmetric center of force, centered at the origin. Assume further that the force is of "finite range" – far enough away, any wave packet acts as if there is no force.[1] In another view, considering the spreading of the wave function, as $t \to \pm\infty$, the wave packet is so spread out that little of it is affected by the force field.

# 2 The $S$ Matrix

Let $\phi(\mathbf{p}, t)$ be the momentum space wave function of the scattering wave packet. Assume it is normalized: $\langle\phi|\phi\rangle = 1$. The time dependent Schrödinger equation is:

$$i\partial_t \phi(\mathbf{p}, t) = H\phi(\mathbf{p}, t), \tag{1}$$

where $H$ includes the influence of the force. Let $E(\mathbf{p}) = \sqrt{m^2 + p^2}$, or $p^2/2m$ in the non-relativistic case, where $p = |\mathbf{p}|$. We restrict consideration to the subspace in which $\phi(\mathbf{p}, t)$ is orthogonal to any bound state wave functions, if any exist. That is, we restrict ourselves to "scattering", in which the wave function can be considered to be far away from the center of force at large times.

Corresponding to $\phi(\mathbf{p}, t)$ (an exact solution to the Schrödinger equation for a possible scattering event), define the notion of an "initial" wave function, $\phi_i(\mathbf{p})$, by taking the following limit at asymptotically early times:

$$\phi_i(\mathbf{p}) = \lim_{t\to-\infty} \phi(\mathbf{p}, t)e^{itE(\mathbf{p})}. \tag{2}$$

[1] The Coulomb force is thus excluded by this criterion. As seen in the exercises of the Approximate Methods note, the cross section for the Coulomb interaction is infinite. However, we have also seen how the Coulomb force may be treated as a limiting case.

That is, $\phi_i(\mathbf{p})e^{-itE(\mathbf{p})}$ is a solution to the Schrödinger equation in the absence of the scattering force. Likewise, we define a "final" wave function according to:

$$\phi_f(\mathbf{p}) = \lim_{t\to\infty} \phi(\mathbf{p}, t)e^{itE(\mathbf{p})}. \tag{3}$$

We should perhaps remark a bit further on the choice of phase factor in these asymptotic limits. Put most simply, these choices just correspond to the interaction picture. However, we may gain a bit of additional insight as follows: Recall that for a free particle the momentum space wave function behaves under a time translation by $t_0$ as:

$$M(t_0)\psi(\mathbf{p}, t) = e^{it_0\frac{p^2}{2m}}\psi(\mathbf{p}, t). \tag{4}$$

Thus, our initial asymptotic limit, Eqn. 2, is like taking

$$\phi_i(\mathbf{p}) = M(t_0 \to -\infty)\phi(\mathbf{p}, t = 0), \tag{5}$$

*if* the motion were strictly force-free for all times. A similar correspondence holds for the asymptotic final state wave function. Thus, our initial and final phases are "matched" to the $t = 0$ phase for the case of no force. Of course, even with a force, $\phi_i(\mathbf{p})$ determines $\phi(\mathbf{p}, t)$, and hence $\phi_f(\mathbf{p})$. We note that it is convenient to deal with momentum space wave functions in this discussion of scattering, because the coordinate space wave functions in general become infinitely "spread out" as $t \to \infty$.

We are couching our discussion in terms of physically realizable wave packets here. Typically, textbooks resort to the use of ("unphysical") plane-wave states, which simplifies the treatment somewhat, at the expense of requiring further justification. It is reassuring to find that we can obtain the same results by considering only physical states. Thus, our development follows closely the nice pedagogical treatment in Ref. [1].

We are interested in the problem of determining $\phi_f$, given $\phi_i$. In principle, this requires solving the Schrödinger equation with $\phi_i$ specifying the initial boundary condition. Usually, however, we do not require the explicit detailed solution – we are interested in only certain aspects of the relation between $\phi_i$ and $\phi_f$, such as the force-induced phase difference.

The transformation giving $\phi_f$, in terms of $\phi_i$ is a linear transformation, which also preserves the normalization of the wave packet. Hence, it is a unitary transformation:

$$|\phi_f\rangle = S|\phi_i\rangle, \tag{6}$$

where

$$SS^\dagger = I. \tag{7}$$

The unitary operator $S$ is called the **S matrix**.

In general, we may represent the $S$ matrix as an integral transform:

$$\phi_f(\mathbf{p}) = \int_{(\infty)} d^3(\mathbf{q}) S(\mathbf{p},\mathbf{q})\phi_i(\mathbf{q}), \tag{8}$$

and the unitarity condition is represented as:

$$\begin{aligned}\int_{(\infty)} d^3(\mathbf{q}) S(\mathbf{p}',\mathbf{q}) S^\dagger(\mathbf{q},\mathbf{p}'') &= \int_{(\infty)} d^3(\mathbf{q}) S(\mathbf{p}',\mathbf{q}) S^*(\mathbf{p}'',\mathbf{q}) \\ &= \int_{(\infty)} d^3(\mathbf{q}) S^*(\mathbf{q},\mathbf{p}') S(\mathbf{q},\mathbf{p}'') \\ &= \delta^{(3)}(\mathbf{p}'-\mathbf{p}''). \end{aligned} \tag{9}$$

The scattering matrix $S(\mathbf{p},\mathbf{q})$ contains precisely all the information which is available in an "asymptotic" experiment, as described by $\phi_i$, $\phi_f$, where the details of what happens near the origin are not observed.

Consider briefly the classical analog, for the classical scattering of a particle of mass $m$ on a static, spherically symmetric force centered at the origin. The motion is described by canonical variables $\mathbf{x}(t)$ and $\mathbf{p}(t)$, which may be obtained by solving the Hamilton equations of motion:

$$\dot{x}_i = \partial_{p_i} H \tag{10}$$

$$\dot{p}_i = -\partial_{x_i} H. \tag{11}$$

The quantities $\mathbf{x}(t)$ and $\mathbf{p}(t)$ are the classical analog of $\phi(\mathbf{p},t)$, since they describe the detailed behavior of the particle. As long as the force falls off sufficiently rapidly at large distances, the motion at large distances and times (presuming no bound states are involved) is uniform:

$$\mathbf{x}(t) = \begin{cases} \mathbf{x}_i + \mathbf{p}_i t/m, & \text{as } t \to -\infty, \\ \mathbf{x}_f + \mathbf{p}_f t/m, & \text{as } t \to \infty. \end{cases} \tag{12}$$

Thus, we can define the "initial" and "final" asymptotic values as:

$$\mathbf{p}_i = \lim_{t\to-\infty} \mathbf{p}(t), \quad \mathbf{x}_i = \lim_{t\to-\infty}\left[\mathbf{x}(t) - \frac{t}{m}\mathbf{p}_i\right] \tag{13}$$

$$\mathbf{p}_f = \lim_{t\to\infty} \mathbf{p}(t), \quad \mathbf{x}_f = \lim_{t\to\infty}\left[\mathbf{x}(t) - \frac{t}{m}\mathbf{p}_f\right]. \tag{14}$$

In particular, $\mathbf{x}_i$, $\mathbf{x}_f$ are (or rather, is) the positions that would occur at $t = 0$ *if* the motion were strictly uniform for all times, with the asymptotic momenta. Thus, $(\mathbf{x}_i, \mathbf{p}_i)$ is the analog of $\phi_i(\mathbf{p})$, and $(\mathbf{x}_f, \mathbf{p}_f)$ is the analog of $\phi_f(\mathbf{p})$.

Once again, specification of $(\mathbf{x}_i, \mathbf{p}_i)$ uniquely determines $(\mathbf{x}_f, \mathbf{p}_f)$, and the transformation giving this relationship is the analog of the $S$ matrix. Note that, in the absence of a force field,

$$\mathbf{x}_i = \mathbf{x}_f, \quad \mathbf{p}_i = \mathbf{p}_f. \tag{15}$$

The analogous statement in quantum mechanics is

$$\phi_i(\mathbf{p}) = \phi_f(\mathbf{p}), \tag{16}$$

since $\phi(\mathbf{p}, t) = \phi_i(\mathbf{p})e^{-itE(\mathbf{p})}$ is the solution of the Schrödinger equation for free particle motion:

$$\begin{aligned} \phi_f(\mathbf{p}) &= \lim_{t\to\infty} \phi(\mathbf{p}, t)e^{itE(\mathbf{p})} && (17) \\ &= \lim_{t\to\infty} \phi_i(\mathbf{p})e^{-itE(\mathbf{p})}e^{itE(\mathbf{p})}, \text{ (free particle)} && (18) \\ &= \phi_i(\mathbf{p}). && (19) \end{aligned}$$

# 3 The Differential Cross Section

In the Approximate Methods note, we defined a **differential cross section** in the discussion of time-dependent perturbation theory. We now return to this, and develop the notion of a cross section with more care. We start with the $S$ matrix formalism just introduced. Given $\phi_i(\mathbf{p})$ and $S$, we obtain $\phi_f(\mathbf{p})$ according to:

$$\phi_f(\mathbf{p}) = \int_{(\infty)} d^3(\mathbf{q})S(\mathbf{p}, \mathbf{q})\phi_i(\mathbf{q}). \tag{20}$$

Now define the **scattered wave** by:

$$\phi_s(\mathbf{p}) = \phi_f(\mathbf{p}) - \phi_i(\mathbf{p}), \tag{21}$$

or

$$\begin{aligned} |\phi_s\rangle &= |\phi_f\rangle - |\phi_i\rangle && (22) \\ &= (S - I)|\phi_i\rangle. && (23) \end{aligned}$$

In the absence of a force, $\phi_f(\mathbf{p}) = \phi_i(\mathbf{p})$, hence $S = I$ and $\phi_s(\mathbf{p}) = 0$. Thus, the name "scattered wave" is appropriate for $\phi_s(\mathbf{p})$.

Since the scattering force is assumed to be stationary, energy is conserved in the scattering event (*e.g.*, for an infinitely heavy source of force, any finite momentum transfer to the source yields zero kinetic energy transfer). Hence, the asymptotic initial and final magnitudes of momenta will be equal. We may thus write:

$$S(\mathbf{p},\mathbf{q}) - \delta^{(3)}(\mathbf{p}-\mathbf{q}) = \delta(p-q)T(\mathbf{p},\mathbf{q}), \tag{24}$$

where $p = |\mathbf{p}|$, $q = |\mathbf{q}|$, and we have extracted the factor $\delta(p-q)$ from the integral transform for $S - I$. Thus, we have:

$$\phi_s(\mathbf{p}) = \int_{(\infty)} d^3(\mathbf{q})\delta(p-q)T(\mathbf{p},\mathbf{q})\phi_i(\mathbf{q}). \tag{25}$$

Note that $T(\mathbf{p},\mathbf{q})$ is "physical" only for $p = q$ ("on the energy shell"), though it is sometimes useful to extend its domain of definition analytically "off-shell".

Let $\{\phi_0(\mathbf{p};\alpha)\,|\,\alpha \in$ some index set$\}$ be a set of potential wave packets, such as might be available in a given experimental arrangement (*e.g.*, a beam delivered by an accelerator). We assume that the particles in our "beam" all have approximately the same momentum $\mathbf{p}_i$, *i.e.*,

$$\phi_0(\mathbf{p};\alpha) \approx 0, \quad \text{unless } \mathbf{p} \approx \mathbf{p}_i. \tag{26}$$

Assume further that $\langle\phi_0|\phi_0\rangle = 1$ for all $\alpha$, and that $\mathbf{p}_i$ is in the 3-direction: $\mathbf{p}_i = p_i\hat{\mathbf{e}}_3$. The index $\alpha$ is a "shape parameter", since we use it to specify the shape of the incident wave packet of a particle in the beam. note that different particles may have different wave packets. We consider the packet

$$\phi_i(\mathbf{p};\alpha;\mathbf{x}) = \phi_0(\mathbf{p};\alpha)e^{-i\mathbf{x}\cdot\mathbf{p}}, \tag{27}$$

obtained by translating the packet $\phi_0(\mathbf{p};\alpha)$ by amount $\mathbf{x}$ in space (assumed to be transverse to the three direction, $x_3 = 0$). Fixing $\mathbf{x}$ and $\alpha$, and taking the packet as our initial wave, we obtain the scattered wave as:

$$\phi_s(\mathbf{p};\alpha;\mathbf{x}) = \int_{(\infty)} d^3(\mathbf{q})\delta(p-q)T(\mathbf{p},\mathbf{q})\phi_0(\mathbf{q};\alpha)e^{-i\mathbf{x}\cdot\mathbf{q}}. \tag{28}$$

We are interested in the probability, $P(\mathbf{u};\alpha;\mathbf{x})d\Omega_u$, that the particle is in the scattered wave with momentum in $d\Omega_u$ around unit vector $\mathbf{u}$. At least if

$\mathbf{u}$ is not too close to the forward direction defined by $\mathbf{p}_i$, this probability is just the probability that the particle's momentum after scattering is in $d\Omega_u$:

$$
\begin{aligned}
P(\mathbf{u};\alpha;\mathbf{x}) &= \int_0^\infty p^2 dp \left|\phi_s(p\mathbf{u};\alpha;\mathbf{x})\right|^2 && (29)\\
&= \int_0^\infty p^2 dp \int_{(\infty)} d^3(\mathbf{q}) \int_{(\infty)} d^3(\mathbf{q}')\delta(p-q)\delta(p-q') && (30)\\
&\quad T(p\mathbf{u},\mathbf{q})T^*(p\mathbf{u},\mathbf{q}')\phi_0(\mathbf{q};\alpha)\phi_0^*(\mathbf{q}';\alpha)e^{-i\mathbf{x}\cdot(\mathbf{q}-\mathbf{q}')}. && (31)
\end{aligned}
$$

We may integrate over $p$, setting $p = q'$:

$$
\begin{aligned}
P(\mathbf{u};\alpha;\mathbf{x}) &= \int_{(\infty)} d^3(\mathbf{q}) \int_{(\infty)} d^3(\mathbf{q}')q^2\delta(q-q') && (32)\\
&\quad T(q\mathbf{u},\mathbf{q})T^*(q'\mathbf{u},\mathbf{q}')\phi_0(\mathbf{q};\alpha)\phi_0^*(\mathbf{q}';\alpha)e^{-i\mathbf{x}\cdot(\mathbf{q}-\mathbf{q}')}, && (33)
\end{aligned}
$$

where we have made use of the fact that $q = q'$ for non-vanishing contributions.

The point of the parameters $\alpha, \mathbf{x}$ is the following: We wish to make sure that our formalism is valid for real experimental situations, *i.e.*, that we have not left out some essential feature in our theory for describing experimental situations. Thus, we assume that the real exdperimental "beam" is a statistical ensemble of wave packets $\phi_0(\mathbf{p};\alpha;\mathbf{x})$. Thus, the parameter $\mathbf{x}$ describes the displacement of a wave packet in this ensemble from some "ideal" (*i.e.*, $\mathbf{x} = 0$) position, and $\alpha$ describes the shape of the wave packet in the ensemble of shapes. The ensemble is described by specifying the probability distributions for $\mathbf{x}$ and $\alpha$, which depends on the particular experimental arrangement. Note that we are interested here in scattering of a "one particle beam" on a "one particle target", *i.e.*, correlations among particles in the beam or in the target are assumed to be zero.

Let $P(\mathbf{u})d\Omega_u$ be the probability that a beam particle comes out in $d\Omega_u$ about $\mathbf{u}$. We compute $P(\mathbf{u})$ by averaging $P(\mathbf{u};\alpha;\mathbf{x})$ over the ensemble probability distribution:

$$
P(\mathbf{u}) = \int_{\{\alpha\}} f(\alpha)d\alpha \frac{1}{\pi R^2} \int_{|\mathbf{x}|\leq \mathbf{R}} d^2(\mathbf{x})P(\mathbf{u};\alpha;\mathbf{x}), \qquad (34)
$$

where $f(\alpha)$ is the probability distribution for $\alpha$, with

$$
\int_{\{\alpha\}} f(\alpha)d\alpha = 1. \qquad (35)
$$

The other integral is evaluated on a circular disk centered on the origin and perpendicular to the beam direction. We have assumed here that the probability distribution for $\mathbf{x}$ is uniform over the disk, and that the size of the disk (or, size of the uniform region of the beam) is large compared with relevant target dimensions (*e.g.*, atomic or nuclear sizes). These assumptions may be modified as the situation warrants.

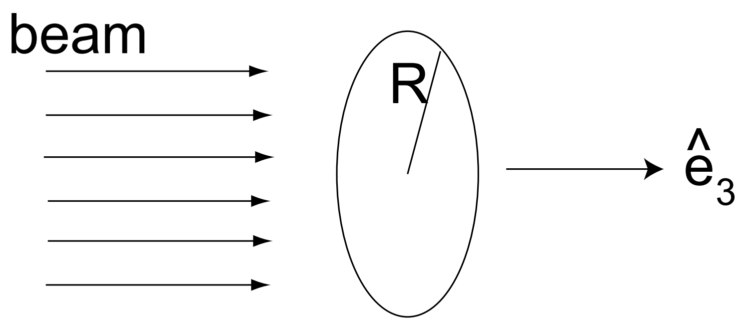


Figure 1: Illustration of a beam setup, with uniformity over a disk of radius $R$.

We define an "effective" differential cross section by:

$$
\begin{aligned}
d\sigma_{\text{eff}}(\mathbf{u}) &= \pi R^2 P(\mathbf{u}) d\Omega_u && (36)\\
&= \int_{\{\alpha\}} f(\alpha) d\alpha \int_{|\mathbf{x}|\leq \mathbf{R}} d^2(\mathbf{x}) P(\mathbf{u};\alpha;\mathbf{x}) d\Omega_u. && (37)
\end{aligned}
$$

This cross section is the differential cross section, per scattering center, which we would observe experimentally with our given beam (*e.g.*, the observed event rate for a scattering is equal to $d\sigma_{\text{eff}}$ times the "luminosity"). It depends on the properties of our beam (*i.e.*, the probability distributions for $\alpha$ and $\mathbf{x}$) and hence, is an "effective" cross section rather than a "fundamental" cross section depending only on the interaction. We wish to define such a fundamental cross section and relate it to the $S$-matrix.

Consider the limit in which $R \to \infty$, and the momentum in the beam is sharply defined. First, for $R \to \infty$:

$$
\begin{aligned}
d\sigma_{\text{eff}}(\mathbf{u}) &= \int_{\{\alpha\}} f(\alpha) d\alpha \int_{(\infty)} d^2(\mathbf{x}) P(\mathbf{u};\alpha;\mathbf{x}) && (38)\\
&= \int_{\{\alpha\}} f(\alpha) d\alpha \int_{(\infty)} d^2(\mathbf{x}) \int_{(\infty)} d^3(\mathbf{q}) \int_{(\infty)} d^3(\mathbf{q}') q^2 \delta(q-q') && (39)
\end{aligned}
$$

$$T(q\mathbf{u},\mathbf{q})T^*(q'\mathbf{u},\mathbf{q}')\phi_0(\mathbf{q};\alpha)\phi_0^*(\mathbf{q}';\alpha)e^{-i\mathbf{x}\cdot(\mathbf{q}-\mathbf{q}')}. \tag{40}$$

Noting that

$$\int_{(\infty)} d^2(\mathbf{x})e^{-i\mathbf{x}\cdot(\mathbf{q}-\mathbf{q}')} = (2\pi)^2\delta(q_1-q_1')\delta(q_2-q_2'), \tag{41}$$

we have:

$$\begin{aligned} d\sigma_{\text{eff}}(\mathbf{u}) &= \int_{\{\alpha\}} f(\alpha)d\alpha \int_{(\infty)} d^3(\mathbf{q}) \int_{(\infty)} d^3(\mathbf{q}')q^2\delta(q_1-q_1')\delta(q_2-q_2')\delta(q-q') \\ &\quad (2\pi)^2 T(q\mathbf{u},\mathbf{q})T^*(q'\mathbf{u},\mathbf{q}')\phi_0(\mathbf{q};\alpha)\phi_0^*(\mathbf{q}';\alpha). \end{aligned} \tag{42}$$

We next use the identity:

$$\delta(q_1-q_1')\delta(q_2-q_2')\delta(q-q') = \frac{q}{|q_3|}\left[\delta^{(3)}(\mathbf{q}-\mathbf{q}') + \delta(q_1-q_1')\delta(q_2-q_2')\delta(q_3+q_3')\right]. \tag{43}$$

While substituting this in, let us also apply the assumption that the momentum in the ("incident") beam is well-defined, *i.e.*,

$$\phi_0(\mathbf{p};\alpha) \approx 0, \quad \text{unless } \mathbf{p} \approx \mathbf{p}_i = p_i\hat{\mathbf{e}}_3. \tag{44}$$

Thus, the product

$$\phi_0(\mathbf{q};\alpha)\phi_0^*(\mathbf{q}';\alpha) \approx 0, \quad \text{unless both } \mathbf{q} \approx \mathbf{p}_i \text{ and } \mathbf{q}' \approx \mathbf{p}_i. \tag{45}$$

In particular, this product is small unless $\mathbf{q} \approx \mathbf{q}'$, and $q_3 \approx q_3'$. Thus, we may neglect the contribution in the integrand from the $\delta(q_3+q_3')$ term, and obtain:

$$\begin{aligned} d\sigma_{\text{eff}}(\mathbf{u}) &= (2\pi)^2 \int_{\{\alpha\}} f(\alpha)d\alpha \int_{(\infty)} d^3(\mathbf{q}) \int_{(\infty)} d^3(\mathbf{q}')\frac{q^3}{|q_3|}\delta^{(3)}(\mathbf{q}-\mathbf{q}') \\ &\quad T(q\mathbf{u},\mathbf{q})T^*(q'\mathbf{u},\mathbf{q}')\phi_0(\mathbf{q};\alpha)\phi_0^*(\mathbf{q}';\alpha) \qquad (46) \\ &= (2\pi)^2 \int_{\{\alpha\}} f(\alpha)d\alpha \int_{(\infty)} d^3(\mathbf{q})\frac{q^3}{|q_3|}|T(q\mathbf{u},\mathbf{q})|^2|\phi_0(\mathbf{q};\alpha)|^2. \qquad (47) \end{aligned}$$

Now, let us again impose our assumption of a well-defined beam momentum, in a more stringent way. Let us require that $|\phi_0(\mathbf{q};\alpha)|^2$ is sufficiently sharply peaked about $\mathbf{q} = \mathbf{p}_i$ that the function

$$\frac{q^3}{|q_3|}|T(q\mathbf{u},\mathbf{q})|^2 \tag{48}$$

does not vary significantly over the region where $|\phi_0(\mathbf{q};\alpha)|^2$ is significant. With this assumption, we may take:

$$\frac{q^3}{|q_3|}|T(q\mathbf{u},\mathbf{q})|^2 \rightarrow \frac{p_i^3}{|p_{i3}|}|T(p_i\mathbf{u},\mathbf{p}_i)|^2 = p_i^2|T(p_i\mathbf{u},\mathbf{p}_i)|^2. \tag{49}$$

Then

$$\begin{aligned} d\sigma_{\text{eff}}(\mathbf{u}) &= (2\pi p_i)^2|T(q\mathbf{u},\mathbf{q})|^2\int_{\{\alpha\}} f(\alpha)d\alpha\int_{(\infty)} d^3(\mathbf{q})|\phi_0(\mathbf{q};\alpha)|^2 & (50)\\ &= (2\pi p_i)^2|T(q\mathbf{u},\mathbf{q})|^2. & (51)\end{aligned}$$

Thus, in this limit, the effective cross section no longer depends on the precise shape of the beam distribution. We interpret this limiting form as the desired "fundamental" (differential) cross section, $\sigma(\mathbf{p}_f;\mathbf{p}_i)$.

We may rewrite the $S$ matrix, as an integral transform, in the form:

$$\begin{aligned} S(\mathbf{p}_f;\mathbf{p}_i) &= \delta^{(3)}(\mathbf{p}_f-\mathbf{p}_i)+\delta(p_f-p_i)T(\mathbf{p}_f;\mathbf{p}_i) & (52)\\ &= \delta^{(3)}(\mathbf{p}_f-\mathbf{p}_i)+\frac{i}{2\pi p_i}\delta(p_f-p_i)f(\mathbf{p}_f,\mathbf{p}_i), & (53)\end{aligned}$$

where we have defined the **scattering amplitude**:

$$f(\mathbf{p}_f,\mathbf{p}_i) \equiv -2\pi i p_i T(\mathbf{p}_f;\mathbf{p}_i). \tag{54}$$

The cross section is given by:

$$\begin{aligned} \sigma(\mathbf{p}_f;\mathbf{p}_i) &= (2\pi p_i)^2|T(\mathbf{p}_f;\mathbf{p}_i)|^2 & (55)\\ &= |f(\mathbf{p}_f,\mathbf{p}_i)|^2. & (56)\end{aligned}$$

That is, the scattering cross section, appropriately, is the square of the scattering amplitude. Note that the scattering amplitude has units of length.

We have defined a differential cross section, and related the $S$ matrix to the scattering amplitude. We have made some assumptions concerning the physical situation, and the results may need to be modified in certain cases. For example, there may be narrow "resonances" where the assumption that the beam is well-defined compared with the variation in $|T|^2$ breaks down, and hence the effective and fundamental cross sections differ (Fig. 2).

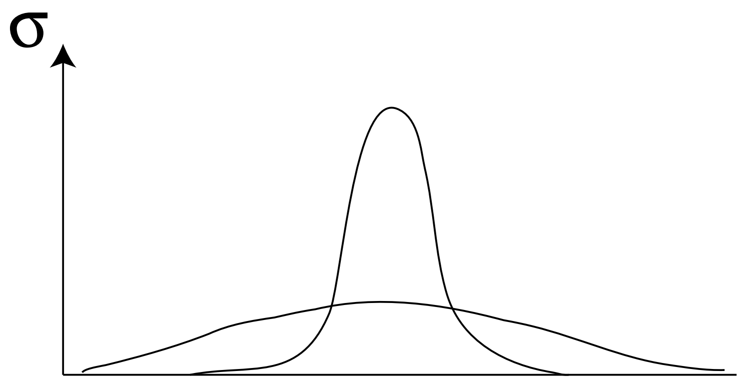


Figure 2: The fundamental cross section (narrow peak) may become smeared out to the effective cross section (wide peak).

# 4 Partial Wave Expansion

For a spherically symmetric center-of-force, the symmetry may be used to obtain a useful expression for the scattering amplitude (and, hence, the $S$ matrix), called the **partial wave expansion**. This form is useful in practice because it is often the case that only a few terms in the expansion contribute significantly to the description of the scattering process. In particular, if the wavelength of the scattered particle is large compared with the "size" of the force center, only a few phase shifts dominate – this is "low energy" scattering.

We may obtain the partial wave expansion as follows: By spherical symmetry, $S(\mathbf{p},\mathbf{q})$ can only depend on rotationally invariant quantities. There are three such invariants in spinless scattering: $\mathbf{p}^2$, $\mathbf{q}^2$ and $\mathbf{p}\cdot\mathbf{q}$. Since $|\mathbf{p}| = |\mathbf{q}|$, we are left with only two distinct invariants. Thus, we may write the $S$ matrix in the form:

$$S(\mathbf{p},\mathbf{q}) = \frac{1}{pq}\delta(p-q)B(p;\mathbf{u}_p\cdot\mathbf{u}_q), \tag{57}$$

where

$$\mathbf{u}_p\cdot\mathbf{u}_q \equiv \frac{\mathbf{p}\cdot\mathbf{q}}{p^2}. \tag{58}$$

Notice that this form is such that $B$ is dimensionless.

Now consider an eigenstate of $\mathbf{J}^2$ and $J_3$ with eigenvalues $j(j+1)$ and $m = 0$ (the initial momentum is along $\hat{\mathbf{e}}_3$, hence $\langle J_3 \rangle = \langle L_3 \rangle = 0$):

$$\phi_{j,0}(\mathbf{p}) = \phi(p) P_j(\mathbf{u}_p \cdot \mathbf{u}_3), \tag{59}$$

where $P_j$ is a Legendre polynomial. Let $S$ operate on this initial wave function:

$$\begin{aligned}
\phi'_{j,0}(\mathbf{p}) &= \int d^3(\mathbf{q}) S(\mathbf{p}, \mathbf{q}) \phi_{j,0}(\mathbf{q}) && (60) \\
&= \int d^3(\mathbf{q}) \frac{1}{pq} \delta(p-q) B(p; \mathbf{u}_p \cdot \mathbf{u}_q) \phi(q) P_j(\mathbf{u}_q \cdot \mathbf{u}_3) && (61) \\
&= \phi(p) \int_{4\pi} d\Omega_{u'} B(p; \mathbf{u}_p \cdot \mathbf{u}') P_j(\mathbf{u}' \cdot \mathbf{u}_3). && (62)
\end{aligned}$$

By conservation of angular momentum, $\phi'_{j,0}(\mathbf{p})$ is an eigenstate of $\mathbf{J}^2$ and $J_3$ with eigenvalues $j(j+1)$ and $m = 0$. Since $S$ is unitary, $\phi'_{j,0}(\mathbf{p})$ is normalized to one if $\phi_{j,0}(\mathbf{p})$ is. Thus, $\phi'_{j,0}(\mathbf{p})/\phi_{j,0}(\mathbf{p})$ must be a function of $p$ of modulus one, and

$$\phi'_{j,0}(\mathbf{p}) = \phi'(p) P_j(\mathbf{u}_p \cdot \mathbf{u}_3). \tag{63}$$

Hence, we may write

$$\begin{aligned}
\phi'_{j,0}(\mathbf{p}) &= e^{2i\delta_j(p)} \phi_{j,0}(\mathbf{p}) && (64) \\
&= e^{2i\delta_j(p)} \phi(p) P_j(\mathbf{u}_p \cdot \mathbf{u}_3), && (65)
\end{aligned}$$

where the **phase shifts**, $\delta_j(p)$, are real functions of $p$. Substituting into Eqn. 62, we have

$$e^{2i\delta_j(p)} \phi(p) P_j(\mathbf{u}_p \cdot \mathbf{u}_3) = \int_{4\pi} d\Omega_{u'} B(p; \mathbf{u}_p \cdot \mathbf{u}') P_j(\mathbf{u}' \cdot \mathbf{u}_3). \tag{66}$$

We obtain an expression for $\delta_j(p)$ by letting $\mathbf{u}_p = \mathbf{u}_3$ and using the fact that $P_j(1) = 1$:

$$\begin{aligned}
e^{2i\delta_j(p)} \phi(p) &= \int_{4\pi} d\Omega_{u'} B(p; \mathbf{u}_3 \cdot \mathbf{u}') P_j(\mathbf{u}' \cdot \mathbf{u}_3) && (67) \\
&= 2\pi \int_{-1}^{1} d\cos\theta B(p, \cos\theta) P_j(\cos\theta). && (68)
\end{aligned}$$

The Legendre polynomials are complete, and obey the orthogonality relation:

$$\int_{-1}^{1} dz P_j(z) P_{j'}(z) = \frac{2}{2j+1} \delta_{jj'}. \tag{69}$$

Hence, we may expand:

$$B(p,\cos\theta) = \sum_{j=0}^{\infty} \frac{2j+1}{4\pi} e^{2i\delta_j(p)} P_j(\cos\theta), \tag{70}$$

or

$$S(\mathbf{p},\mathbf{q}) = \frac{\delta(p-q)}{pq} \sum_{j=0}^{\infty} \frac{2j+1}{4\pi} e^{2i\delta_j(p)} P_j(\cos\theta), \tag{71}$$

where $\cos\theta = \mathbf{u}_p \cdot \mathbf{u}_q$. This result, Eqn. 71, is the **partial wave expansion for the $S$ matrix**.

When all phase shifts $\delta_j = 0$, $S = I$, so we may write:

$$S(\mathbf{p},\mathbf{q}) - \delta^{(3)}(\mathbf{p}-\mathbf{q}) = \frac{\delta(p-q)}{4\pi pq} \sum_{j=0}^{\infty} (2j+1) \left[ e^{2i\delta_j(p)} - 1 \right] P_j(\cos\theta). \tag{72}$$

But we also have,

$$S(\mathbf{p},\mathbf{q}) - \delta^{(3)}(\mathbf{p}-\mathbf{q}) = \frac{i}{2\pi p} \delta(p-q) f(\mathbf{p},\mathbf{q}). \tag{73}$$

Comparing these, we obtain the **partial wave expansion for the scattering amplitude**:

$$f(\mathbf{p},\mathbf{q}) = f(p;\cos\theta) = \frac{1}{2ip} \sum_{j=0}^{\infty} (2j+1) \left[ e^{2i\delta_j(p)} - 1 \right] P_j(\cos\theta). \tag{74}$$

In order to have convergence, we suspect we'll have $\delta_j(p) \to 0$ as $j \to \infty$. The unitarity of the $S$ matrix is contained in the reality of the functions $\delta_j(p)$.

# 5 Optical Theorem

Let us consider more generally than the partial wave expansion the unitarity of the $S$ matrix:

$$\int_{(\infty)} d^3(\mathbf{q}) S(\mathbf{p}',\mathbf{q}) S^*(\mathbf{p}'',\mathbf{q}) = \delta^{(3)}(\mathbf{p}'-\mathbf{p}''). \tag{75}$$

Substitute in

$$S(\mathbf{p},\mathbf{q}) = \delta^{(3)}(\mathbf{p}-\mathbf{q}) + \frac{i}{2\pi p} \delta(p-q) f(\mathbf{p},\mathbf{q}), \tag{76}$$

obtaining

$$\begin{aligned}\delta^{(3)}(\mathbf{p}'-\mathbf{p}'') &= \delta^{(3)}(\mathbf{p}'-\mathbf{p}'') \\ &+\frac{i}{2\pi p'}\delta(p'-p'')f(\mathbf{p}',\mathbf{p}'') - \frac{i}{2\pi p''}\delta(p'-p'')f^*(\mathbf{p}'',\mathbf{p}') \\ &+\frac{1}{4\pi^2 p'p''}\int_{(\infty)} q^2 dq d\Omega_u \delta(p'-q)\delta(p''-q)f(\mathbf{p}',\mathbf{q})f^*(\mathbf{p}'',\mathbf{q}).\end{aligned} \tag{77}$$

This simplifies to

$$-\frac{i}{2\pi}\frac{\delta(p'-p'')}{p'}\left[f(\mathbf{p}',\mathbf{p}'')-f^*(\mathbf{p}'',\mathbf{p}')\right] = \frac{\delta(p'-p'')}{4\pi^2}\int_{(4\pi)} d\Omega_u f(\mathbf{p}',\mathbf{q})f^*(\mathbf{p}'',\mathbf{q}), \tag{78}$$

with $q = p'$. But for a symmetric central force we must have

$$f(\mathbf{p}',\mathbf{p}'') = f(\mathbf{p}'',\mathbf{p}') = f(p';\mathbf{u}'\cdot\mathbf{u}''). \tag{79}$$

Thus, letting $p' = p'' = p$, we find

$$\Im f(p;\mathbf{u}'\cdot\mathbf{u}'') = \frac{p}{4\pi}\int_{4\pi} d\Omega_u f(p;\mathbf{u}'\cdot\mathbf{u})f^*(p;\mathbf{u}''\cdot\mathbf{u}). \tag{80}$$

In particular, if we take $\mathbf{u}' = \mathbf{u}''$, then

$$\Im f(p;1) = \frac{p}{4\pi}\int_{4\pi} d\Omega_u |f(p;\mathbf{u}'\cdot\mathbf{u})|^2 \tag{81}$$

$$= \frac{p}{4\pi}\int_{4\pi} d\Omega_u \sigma(p\mathbf{u},p\mathbf{u}') \tag{82}$$

$$= \frac{p}{4\pi}\sigma_T(p), \tag{83}$$

where $\sigma_T$ is the **total cross section**.

We have just derived what is known as the **optical theorem**:

$$\sigma_T(p) = \frac{4\pi}{p}\Im f(p;1). \tag{84}$$

The total cross section is equal to $4\pi/p$ times the imaginary part of the forward scattering amplitude.

The total cross section may also be readily obtained in terms of the partial wave expansion:

$$\sigma_T(p) = \int_{(4\pi)} \frac{d\sigma(p,\cos\theta)}{d\Omega} d\Omega \tag{85}$$

$$
\begin{aligned}
&= \int_{(4\pi)} |f(p,\cos\theta)|^2 d\Omega && (86)\\
&= 2\pi \int_{-1}^{1} dz \frac{1}{4p^2} \sum_{j=0}^{\infty} \sum_{j'=0}^{\infty} (2j+1)(2j'+1) \left[e^{2i\delta_j(p)} - 1\right] \left[e^{-2i\delta_{j'}(p)} - 1\right] P_j(z) P_{j'}(z) \\
&= \frac{4\pi}{p^2} \sum_{j=0}^{\infty} (2j+1) \left[e^{2i\delta_j(p)} - 1\right] \left[e^{-2i\delta_j(p)} - 1\right] && (87)\\
&= \frac{4\pi}{p^2} \sum_{j=0}^{\infty} (2j+1) \sin^2 \delta_j(p). && (88)
\end{aligned}
$$

It is left as an exercise for the reader to show that this result agrees with the optical theorem.

We conclude this section by remarking that the optical theorem is a rather general consequence of wave scattering in which there is a conservation property akin to the conservation of probability in quantum mechanics. For example, it also holds in the scattering of electromagnetic waves where energy and power flow are conserved. For a discussion, see, for example, J. D. Jackson's text "Classical Electrodynamics".

# 6 Interim Remarks

We pause here to make a few remarks concerning the nature of our results and possible extensions to them:

1. Our discussion has been pretty general, up to assumptions of spherical symmetry, and of the force falling off rapidly enough with distance.

2. In particular, the results are valid whether the particles are relativistic or non-relativistic. We nowhere made any assumptions concerning speed, only using general properties of waves and quantum mechanics. Any "wave equation" suffices, since we only use it in the discussion of asymptotic states.

3. The discussion also applies to the elastic collision of two particles in their center-of-mass system. Note that this system has spherical symmetry in this frame. Relativity is unimportant here as well (though we may need it to transform to a different frame of reference). We may also consider inelastic scattering in the CM frame, as long as we properly formulate our conservation of energy.

4. We may further extend the discussion to the elastic scattering of two particles with any spins in the CM frame. The essential change is that the asymptotic wave function for particles of spin $s$ are $(2s+1)$-component momentum space wave functions, and the scattering amplitude $f(\mathbf{p}_f, \mathbf{p}_i)$ now becomes also a matrix operator in spin space. Ref. [2] develops this theory for scattering of the form $a + b \to c + d$. We have actually already seen the essential aspects in our class discussions of the consequence of angular momentum conservation on the description of scattering scattering in the helicity basis.

Note that we have not yet said much about how to determine the phase shifts $\delta_j(p)$ in a problem of interest – we have only shown that they are the key ingredients in the scattering problem. Our discussion will now turn towards this issue.

# 7 Resonances

Consider the partial wave expansion:

$$f(\mathbf{p}, \mathbf{q}) = f(p; \cos\theta) = \frac{1}{2ip}\sum_{j=0}^{\infty} \frac{2j+1}{4\pi}\left[e^{2i\delta_j(p)} - 1\right] P_j(\cos\theta). \tag{89}$$

We note that

$$\frac{1}{2i}\left[e^{2i\delta_j(p)} - 1\right] = \frac{1}{\cot\delta_j - i}. \tag{90}$$

This function has a maximum magnitude of one whenever

$$\delta_j = \frac{\pi}{2} + n\pi, \quad n = \text{integer}. \tag{91}$$

When this occurs, we are said to have a **resonance** in the $j$th partial wave.

Suppose now that the $j$th channel exhibits a resonance at energy $E = E_0$. Then $\cot\delta_j(E)$ (written as a function of energy) vanishes at $E = E_0$. To describe the scattering in the neighborhood of this resonance, we make an expansion of $\cot\delta_j(E)$ to linear order:

$$\cot\delta_j(E) = -\frac{2}{\Gamma}(E - E_0) + O\left[(E - E_0)^2\right]. \tag{92}$$

Thus, in the neighborhood of a resonance, the contribution of this channel

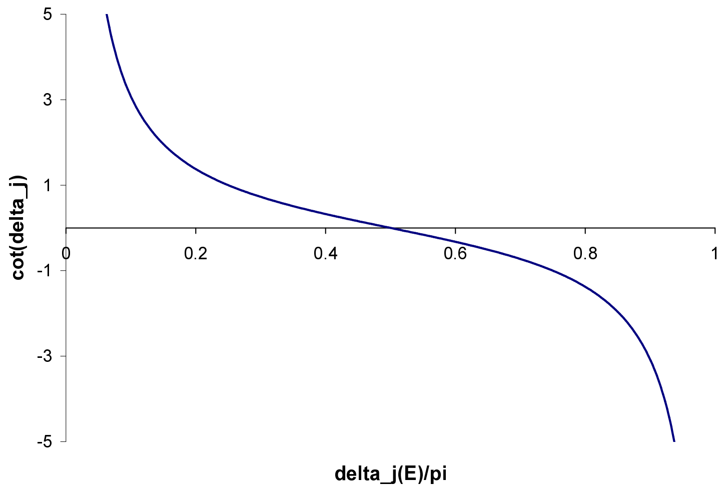


Figure 3: Graph of $\cot \delta_j(E)$, showing linear behavior near $\pi/2$, with negative slope.

to the scattering amplitude is:

$$f_j(p; \cos\theta) = \frac{2j+1}{p} \frac{1}{-\frac{2}{\Gamma}(E-E_0) - i} P_j(\cos\theta) \quad (93)$$

$$= (2j+1) \frac{-\Gamma/2p}{E - E_0 + i\Gamma/2} P_j(\cos\theta). \quad (94)$$

The contribution to the total cross section from this channel alone is (using the optical theorem):

$$\sigma_{Tj}(p) = \frac{4\pi}{p} \Im f_j(p; 1) \quad (95)$$

$$= (2j+1) \frac{4\pi}{p} \Im \frac{-\Gamma/2p}{E - E_0 + i\Gamma/2} \quad (96)$$

$$= -\frac{4\pi(2j+1)}{p^2} \frac{\Gamma}{2} \Im \frac{E - E_0 - i\Gamma/2}{(E-E_0)^2 + \Gamma^2/4} \quad (97)$$

$$= \frac{4\pi}{p^2}(2j+1) \frac{\Gamma^2/4}{(E-E_0)^2 + \Gamma^2/4}. \quad (98)$$

We arrive at a Breit-Wigner form for the cross section in the neighborhood of a resonance. The choice of parameterization is such that $\Gamma$ is the full width at half maximum of the Breit-Wigner distribution. The cross section decreases from its peak value by 1/2 at $E = E_0 \pm \Gamma/2$.

Note, finally, that the maximum contribution to the cross section in partial wave $j$ is just:

$$\sigma_{Tj}(E)\text{max} = \sigma_{Tj}(E_0) = \frac{4\pi}{p^2}(2j+1). \tag{99}$$

This maximum follows from the unitarity bound:

$$\left|\frac{1}{2i}\left[e^{2i\delta_j} - 1\right]\right| = \left|e^{i\delta_j}\sin\delta_j\right| \leq 1. \tag{100}$$

# 8 The Phase Shift

Intuitively, we may think of the phase shift as follows: Suppose, for example, we have a "dominantly" attractive potential. The wave will oscillate more rapidly in the region of the potential than it would if the potential were not present. A wave starting at phase zero at the origin will accumulate phase as it propagates out to large distances. The phase is accumulated faster in the presence of an attractive potential than if no potential is present. Thus, there will be a phase shift of the wave in the potential relative to the wave in no potential at large distances (and once outside the region of the potential, this shift is independent of distance). Fig. 4 provides an illustration of this effect. Asymptotically, the scattered wave will be positively phase shifted with respect to an unscattered wave. Similarly, a dominantly repulsive potential will yield negative phase shifts.

To see how we may get at the phase shifts, let us consider plane wave solutions, and their partial wave expansion. The force-free Schrödinger equation is

$$\nabla^2\psi + k^2\psi = 0, \quad (E = k^2/2m). \tag{101}$$

The plane wave solution for a wave traveling in the $+z$ direction (if $k > 0$) is:

$$\psi(\mathbf{x}) = e^{ikz} = e^{ikr\cos\theta}. \tag{102}$$

The separation of variables expansion of such a function in spherical polar coordinates is:

$$\psi(\mathbf{x}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[A_{\ell m}j_\ell(kr) + B_{\ell m}n_\ell(kr)\right]P_l^m(\theta)e^{im\phi}, \tag{103}$$

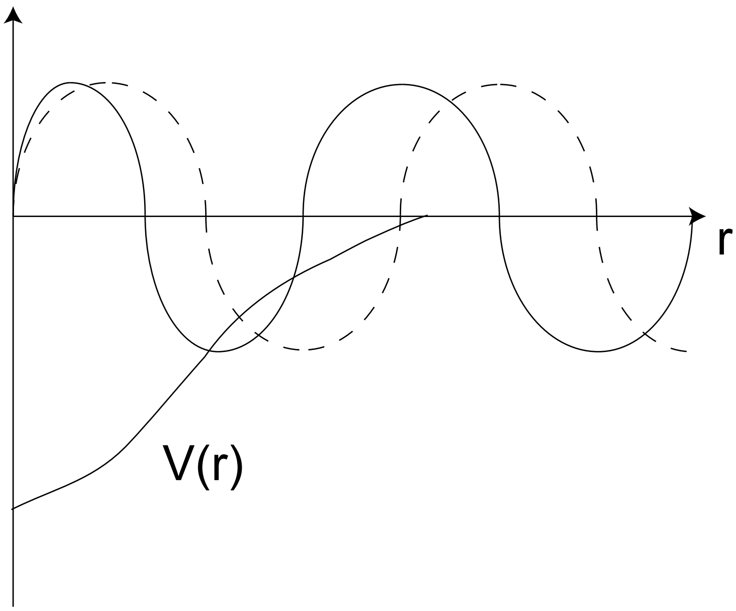


Figure 4: Illustration of the production of a phase shift due to a potential. The dashed line illustrates the wave in the absence of a potential. At large $r$ the accumulated phase (starting from $r = 0$) of the wave in the potential is larger than the accumulated phase of the wave without the potential.

where

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+\frac{1}{2}}(x); \quad n_\ell(x) = (-)^{\ell+1} \sqrt{\frac{\pi}{2x}} J_{-\ell-\frac{1}{2}}(x) \tag{104}$$

are spherical Bessel functions.

Since $e^{ikr\cos\theta}$ does not depend on $\phi$ (beam along the $z$-axis, hence no $z$-component of $L$), only the $m = 0$ terms contribute. Also, $n_\ell(kr)$ is not regular at $r = 0$,[2] so $B_{\ell m} = 0$. Hence,

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} A_\ell j_\ell(kr) P_l(\cos\theta). \tag{107}$$

It remains to determine the coefficients $A_\ell$. We use the orthogonality relation

$$\int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}, \tag{108}$$

to obtain

$$A_\ell j_\ell(kr)\frac{2}{2\ell+1} = \int_{-1}^{1} dx e^{ikrx} P_\ell(x). \tag{109}$$

We can match the two sides by considering large $kr$. For the left hand side,

$$j_\ell(x) \longrightarrow_{x\to\infty} \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right). \tag{110}$$

On the right side, we note that by partial integrations we may obtain an expansion in powers of $1/kr$, *e.g.*, after two such integrations:

$$\begin{aligned}\int_{-1}^{1} dx e^{ikrx} P_\ell(x) &= \frac{1}{ikr} e^{ikrx} P_\ell(x)|_{-1}^{1} \qquad (111)\\ &\quad -\frac{1}{ikr}\left[\frac{1}{ikr} e^{ikrx} P'_\ell(x)|_{-1}^{1} - \frac{1}{ikr}\int_{-1}^{1} dx e^{ikrx} P''_\ell(x)\right].\end{aligned}$$

It appears (left to the reader to prove) that the first term is dominant for large $kr$. Using also $P_\ell(\pm 1) = (\pm)^\ell$, we thus have:

$$\int_{-1}^{1} dx e^{ikrx} P_\ell(x) \to_{kr\to\infty} \frac{1}{ikr}\left[e^{ikr} - (-)^\ell e^{-ikr}\right]. \tag{112}$$

So, we must compare:

$$\begin{aligned}\frac{1}{ikr}\left[e^{ikr} - (-)^\ell e^{-ikr}\right] &= A_\ell \frac{1}{kr}\frac{2}{2\ell+1}\sin\left(kr - \frac{\ell\pi}{2}\right) \qquad (113)\\ &= A_\ell \frac{1}{ikr}\frac{1}{2\ell+1} e^{-i\pi\ell/2}\left[e^{ikr} - (-)^\ell e^{-ikr}\right]. \qquad (114)\end{aligned}$$

---

[2]For small $x$:

$$n_\ell(x) \to_{x\to 0} \begin{cases} -\frac{1}{x}, & \ell = 0 \\ -\frac{(2\ell-1)!!}{x^{\ell+1}}, & \ell > 0; \end{cases} \tag{105}$$

$$j_\ell(x) \to_{x\to 0} \frac{x^\ell}{(2\ell+1)!!}. \tag{106}$$

Solving for $A_\ell = (i)^\ell(2\ell+1)$, and therefore

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}(i)^\ell(2\ell+1)j_\ell(kr)P_l(\cos\theta). \tag{115}$$

Now suppose we have a potential $V(r)$ which vanishes everywhere outside a sphere of radius $R$. The scattering amplitude, in terms of the phase shifts, as we have seen, is:

$$f(k;\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left(e^{2i\delta_\ell}-1\right)P_\ell(\cos\theta). \tag{116}$$

Let us relate this to the scattered wave function. For $r < R$, we may write:

$$\psi = \sum_{\ell=0}^{\infty}(i)^\ell(2\ell+1)R_\ell(k;r)P_\ell(\cos\theta), \tag{117}$$

where $\chi_\ell = rR_\ell$, and

$$\chi_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - 2mV(r)\right]\chi_\ell = 0. \tag{118}$$

Outside the sphere $r = R$ we may write:

$$\psi = \sum_{\ell=0}^{\infty}(i)^\ell(2\ell+1)\left[j_\ell(kr) + \frac{1}{2}\alpha_\ell h_\ell^{(1)}(kr)\right]P_\ell(\cos\theta), \tag{119}$$

where $h_\ell^{(1)}$ is the spherical Hankel function of the first kind:

$$h_\ell^{(1)}(kr) = j_\ell(kr) + in_\ell(kr) \tag{120}$$

$$\rightarrow_{kr\to\infty} \frac{1}{(i)^{\ell+1}}\frac{e^{ikr}}{kr}. \tag{121}$$

The asymptotic form shows that this corresponds to an asymptotically outgoing spherical wave. The spherical Hankel function of the second kind, $h_\ell^{(2)}$, would correspond to an asymptotically incoming spherical wave. We don't include such a term, because we have already included the incoming plane wave in the $j_\ell(kr)$ term.

The asymptotic behavior of the wave function is therefore:

$$\psi \sim \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell\left[\frac{\sin\left(kr-\frac{\ell\pi}{2}\right)}{kr}+\frac{\alpha_\ell}{2}\frac{1}{(i)^{\ell+1}}\frac{e^{ikr}}{kr}\right]P_\ell(\cos\theta) \tag{122}$$

$$\sim \frac{1}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\left[e^{ikr}(1+\alpha_\ell)-(-)^\ell e^{-ikr}\right]P_\ell(\cos\theta). \tag{123}$$

This is now expressed in terms of an incoming spherical wave and an outgoing spherical wave.

For elastic scattering, we must have conservation of the number of particles (unitarity), so

$$|1+\alpha_\ell|^2 = |(-)^\ell|^2, \tag{124}$$

where the left hand side corresponds to the outgoing wave, and the right hand side is the incoming. This is satisfied if $\alpha_\ell$ is of the form:

$$\alpha_\ell = e^{2i\delta_\ell}-1, \tag{125}$$

where $\delta_\ell$ is real.

Now consider the scattered wave function in the asymptotic limit:

$$\psi_S = \psi - e^{ikr\cos\theta} \tag{126}$$

$$= \frac{1}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\left[(1+\alpha_\ell)e^{ikr}-(-)^\ell e^{-ikr}\right]P_\ell(\cos\theta)$$

$$-\sum_{\ell=0}^{\infty}(2\ell+1)(i)^\ell\frac{1}{kr}\sin\left(kr-\frac{\ell\pi}{2}\right)P_\ell(\cos\theta)$$

$$= \frac{1}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\left[(1+\alpha_\ell)e^{ikr}-(-)^\ell e^{-ikr}-e^{ikr}+(-)^\ell e^{-ikr}\right]P_\ell(\cos\theta) \tag{127}$$

$$= \frac{1}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\alpha_\ell e^{ikr}P_\ell(\cos\theta) \tag{128}$$

$$= f(k;\theta)\frac{e^{ikr}}{r}. \tag{129}$$

We see that the scattering amplitude has the interpretation as the coefficient of the outgoing (scattered) spherical wave.

Notice that the outgoing part of the wave in Eqn. 127 is

$$\left[e^{ikr\cos\theta}\right]_{\text{out}} \sim \frac{1}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)e^{ikr}P_\ell(\cos\theta), \tag{130}$$

while the outgoing part of the actual wave (which includes in addition the outgoing part of the unscattered wave) is

$$\psi_{\text{out}} \sim_{kr\to\infty} \frac{1}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)e^{2i\delta_\ell}e^{ikr}P_\ell(\cos\theta), \tag{131}$$

where $e^{2i\delta_\ell} = 1+\alpha_\ell$. Comparing Eqns. 130 and 131, we see that $2\delta_\ell$ is the difference in phase between the outgoing parts of the actual wave function and the $e^{ikr\cos\theta}$ plane wave.

It is perhaps useful to make a comment here concering interference among partial waves. Note that in the differential cross section,

$$\frac{d\sigma}{d\Omega} = |f(k;\theta)|^2, \tag{132}$$

the different partial waves can interfere, *i.e.*, in general we cannot distinguish which partial angular momentum states contribute to the scattering in a given direction. On the other hand, in the total cross section:

$$\sigma_T = \int_{(4\pi)} d\Omega |f(k;\theta)|^2 = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell; \tag{133}$$

by virtue of the orthogonality of the $P_\ell(\cos\theta)$, there is no interference among the partial waves. The cross section decomposes into a sum of partial cross sections of definite angular momenta.

## 8.1 Relation of Phase Shift to Logarithmic Derivative of the Radial Wave Function

Recall the radial wave equation for $r < R$:

$$\chi_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - 2mV(r)\right]\chi_\ell = 0, \quad \chi_\ell = rR_\ell. \tag{134}$$

We must have continuity of the wave function and its derivative on the sphere $r = R$. Hence,

$$R_\ell(k, R) = j_\ell(kR) + \frac{1}{2}\alpha_\ell h_\ell^{(1)}(kR) \quad (135)$$

$$\left.\frac{dR_\ell(k, r)}{dr}\right|_{r=R} = k\left[j'_\ell(kR) + \frac{1}{2}\alpha_\ell h_\ell^{(1)\prime}(kR)\right]. \quad (136)$$

The factor of $k$ on the right side of Eqn: 136 is because the prime notation denotes derivatives with respect to the argument of the function, *i.e.*, with respect to $kr$.

We divide Eqn. 136 by Eqn. 135, and let $x = kR$:

$$\left.\frac{\frac{dR_\ell(k,r)}{dr}}{R_\ell(k, r)}\right|_{r=R} = k\frac{\left[j'_\ell(x) + \frac{1}{2}\alpha_\ell h_\ell^{(1)\prime}(x)\right]}{j_\ell(x) + \frac{1}{2}\alpha_\ell h_\ell^{(1)}(x)}. \quad (137)$$

With $dr = r d\log r$, we may write this in the form:

$$L_\ell \equiv \left.\frac{d\log R_\ell(k, r)}{d\log r}\right|_{r=R} = x\frac{\left[j'_\ell(x) + \frac{1}{2}\alpha_\ell h_\ell^{(1)\prime}(x)\right]}{j_\ell(x) + \frac{1}{2}\alpha_\ell h_\ell^{(1)}(x)}. \quad (138)$$

Solving for $\alpha_\ell = e^{2i\delta_\ell} - 1$:

$$\alpha_\ell = -2\frac{L_\ell j_\ell(x) - x j'_\ell(x)}{L_\ell h_\ell^{(1)}(x) - x h_\ell^{(1)\prime}}, \quad (139)$$

where $x = kR$. Thus, if the logarithmic derivative $L_\ell$ is determined, then the phase shift is known.

## 8.2 Low Energy Limit

Let us use the result just obtained to demonstrate that the partial wave expansion normally converges faster in the low energy limit, $x = kR \ll 1$. We'll start with the power series expansions for the spherical Bessel/Hankel functions:

$$h_\ell^{(1)}(x) = j_\ell(x) + i n_\ell(x), \quad (140)$$

$$j_\ell(x) = 2^\ell x^\ell \sum_{n=0}^{\infty} \frac{(-)^n (n+\ell)!}{n!(2n+2\ell+1)!} x^{2n}, \quad (141)$$

$$n_\ell(x) = (-)^{\ell+1}\frac{1}{2^\ell x^{\ell+1}} \sum_{n=0}^{\infty} \frac{(-)^{n-\ell}(n-\ell)!}{n!(2n-2\ell)!} x^{2n}. \quad (142)$$

As a technical aside, we see that we really need to make sense of the last formula above in the case $\ell > n$. Guided by the identity:

$$z!(-z)! = \frac{\pi z}{\sin \pi z}, \tag{143}$$

we define

$$\frac{(n-\ell)!}{(2n-2\ell)!} = (-)^{n-\ell}\frac{(2\ell-2n)!}{(\ell-n)!}, \quad \text{if } n < \ell. \tag{144}$$

For $x \ll 1$ (the low energy limit), we keep only the $n = 0$ term in these expansions:

$$\begin{aligned} j_\ell(x) &= \frac{\ell!}{(2\ell+1)!}2^\ell x^\ell + O(x^{\ell+1}) &(145)\\ n_\ell(x) &= \frac{(-)^{\ell+1}}{2^\ell x^{\ell+1}}\frac{(-)^{-\ell}(2\ell)!}{\ell!} + O(x^{-\ell+1}). &(146)\end{aligned}$$

Thus,

$$\begin{aligned} L_\ell j_\ell(x) - x j_\ell'(x) &= 2^\ell \frac{\ell!}{(2\ell+1)!}(L_\ell - \ell)x^\ell + O(x^{\ell+2}) &(147)\\ L_\ell h_\ell^{(1)}(x) - x h_\ell^{(1)\prime} &= i\left[L_\ell n_\ell(x) - x n_\ell'(x)\right] + O(x^\ell) &(148)\\ &= -i\frac{(2\ell)!}{2^\ell \ell!}(L_\ell + \ell + 1)\frac{1}{x^{\ell+1}} + O(x^\ell, x^{-\ell+1}). &(149)\end{aligned}$$

Therefore,

$$\alpha_\ell \approx -\frac{2i}{2\ell+1}\left[\frac{2^\ell \ell!}{(2\ell)!}\right]^2 \frac{L_\ell - \ell}{L_\ell + \ell + 1}x^{2\ell+1}. \tag{150}$$

Let us use this result to check the convergence of the partial wave expansion for small $x$. The coefficient of the $P_{\ell+1}(\cos\theta)$ term divided by the coefficient of the $P_\ell(\cos\theta)$ term is:

$$\frac{(2\ell+3)\alpha_{\ell+1}}{(2\ell+1)\alpha_\ell} = \frac{1}{(2\ell+1)^2}\frac{L_{\ell+1} - \ell - 1}{L_{\ell+1} + \ell + 2}\frac{L_\ell + \ell + 1}{L_\ell - \ell}x^2. \tag{151}$$

In general then, we should have very good convergence for $x \ll 1$.

In the low energy limit the $\ell^{\text{th}}$ scattering partial wave is proportional to:

$$\alpha_\ell \propto k^{2\ell+1}. \tag{152}$$

That is,

$$\alpha_\ell = e^{2i\delta_\ell} - 1 = 2i \sin\delta_\ell e^{i\delta_\ell}, \tag{153}$$

or

$$|\sin\delta_\ell| \sim k^{2\ell+1}, \quad \text{as } kR \to 0. \tag{154}$$

Using

$$j_\ell(x) = \frac{1}{2}\left[h_\ell^{(1)}(x) + h_\ell^{(2)}(x)\right], \tag{155}$$

we may also write:

$$1 + \alpha_\ell = -\frac{L_\ell h_\ell^{(2)}(x) - x h_\ell^{(2)\prime}(x)}{L_\ell h_\ell^{(1)}(x) - x h_\ell^{(1)\prime}(x)}. \tag{156}$$

We may obtain exact expressions for the two lowest phase shifts with,

$$h_\ell^{(2)}(x) = h_\ell^{(1)*}(x), \quad \text{for real } x, \tag{157}$$

$$h_0^{(1)}(x) = -\frac{i}{x}e^{ix}, \tag{158}$$

$$h_1^{(1)}(x) = -\frac{1}{x}e^{ix}\left(1 + \frac{i}{x}\right). \tag{159}$$

Hence,

$$1 + \alpha_0 = e^{-2ix}\frac{1 + ix/\hat{L}_0}{1 - ix/\hat{L}_0} \tag{160}$$

$$1 + \alpha_1 = e^{-2ix}\frac{1 + ix - x^2/(\hat{L}_1 + 1)}{1 - ix - x^2/(\hat{L}_1 + 1)}, \tag{161}$$

where $x = kR$ and

$$\hat{L}_\ell \equiv \left.\frac{d\log\chi_\ell}{d\log r}\right|_{r=R}, \tag{162}$$

$$= \left.\frac{d\log R_\ell}{d\log r}\right|_{r=R} + 1. \tag{163}$$

## 8.3 Example

Suppose we scatter from the potential (see Fig. 5):

$$V(r) = \begin{cases} V_0 > 0 & r < R, \\ 0, & r \geq R. \end{cases} \tag{164}$$

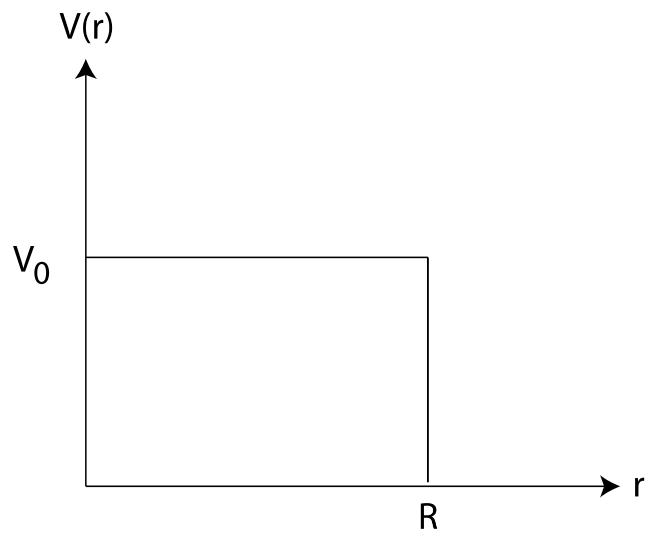


Figure 5: Graph of the example scattering potential.

We wish to determine, for example, the phase shift and partial cross section for $\ell = 0$ (S-wave) scattering. Since the potential is repulsive, we should find $\delta_0 < 0$. Let

$$k^2 = 2mE \tag{165}$$

$$k_0^2 = 2mV_0 \tag{166}$$

$$\Delta = \sqrt{k_0^2 - k^2}. \tag{167}$$

The radial wave equation for $r < R$ is

$$\chi_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - k_0^2\right]\chi_\ell = 0. \tag{168}$$

In particular, for $\ell = 0$:[3]

$$\chi_0'' + (k^2 - k_0^2)\chi_0 = 0. \tag{169}$$

[3]Recall that we may express the wave function for the spherically-symmetric central force problem in general in the form

$$\psi(\mathbf{x}) = \sum_{n,\ell,m} A_{n\ell m} R_{n\ell}(r) Y_{\ell m}(\theta, \phi),$$

If $E < V_0$, then $\Delta > 0$, and $\chi_0'' = \Delta^2\chi_0$, or

$$\chi_0 = A\sinh\Delta r, \quad r < R,\ E < V_0. \tag{170}$$

If instead $E > V_0$, then $\Delta = i\Delta'$ is purely imaginary, hence

$$\chi_0 = A'\sin\Delta' r, \quad r < R,\ E > V_0. \tag{171}$$

Outside the sphere bounded by $r = R$, we have $\chi_0'' = -k^2\chi_0$, hence

$$\chi_0(r) = \sin(kr + \delta), \quad r > R \tag{172}$$

where we have chosen an arbitrary normalization. We compare this with our expansion:

$$\psi(\mathbf{x}) = \sum_{\ell=0}^{\infty} i^\ell(2\ell+1)\left[j_\ell(kr) + \frac{1}{2}\alpha_\ell H_\ell^{(1)}(kr)\right]P_\ell(\cos\theta), \quad r > R. \tag{173}$$

Consider the $\ell = 0$ term in particular:

$$\chi_0(r) = r\left[j_0(kr) + \frac{1}{2}\alpha_0 h_0^{(1)}(kr)\right]. \tag{174}$$

Using, $j_0(x) = \sin x/x$, $h_0^{(1)}(x) = -ie^{ix}/x$, and $\alpha_0 = e^{2i\delta_0} - 1$, we obtain (not worrying about the overall normalization),

$$\chi_0(r) = \sin kr - \frac{i}{2}\alpha_0 e^{ikr} \tag{175}$$

$$= \frac{1}{2i}\left(e^{ikr} - e^{-ikr} + e^{ikr+2i\delta_0} - e^{ikr}\right) \tag{176}$$

$$= e^{i\delta_0}\sin(kr + \delta_0), \tag{177}$$

or, simply $\chi_0(r) = \sin(kr + \delta_0)$, absorbing the phase factor into the normaliztion. Comparing with Eqn. 172, we find $\delta = \delta_0$.

---

where

$$\frac{1}{2mr}\frac{\partial^2}{\partial r^2}\left[rR_{n\ell}(r)\right] + \left[\frac{\ell(\ell+1)}{2mr^2} - V(r)\right]R_{n\ell}(r) = E_n R_{n\ell}(r)$$

($n$ may be a continuous index in general), and we have defined $\chi_{n\ell}(r) \equiv rR_{n\ell}(r)$. At least as long as $V(r)$ is finite as $r \to 0$, $R_{n\ell}(0)$ must be finite, hence the boundary condition $\chi_{n\ell}(0) = 0$.

We determine $\delta_0$ by matching $\chi_0$ and $\chi_0'$ at $r = R$. The logarithmic derivative $L_0 = R\chi_0'(R)/\chi_0(R)$ contains the information about the parameter of interest, $\delta_0$, and we need never determine $A$ (or $A'$). For $E < V_0$:

$$L_0 = \Delta R \frac{\cosh \Delta R}{\sinh \Delta R} = \Delta R \coth \Delta R \tag{178}$$

$$= kR \frac{\cos(kR + \delta_0)}{\sin(kR + \delta_0)} = kR \cot(kR + \delta_0). \tag{179}$$

That is,

$$L_0 = kR \cot(kR + \delta_0) = \Delta R \coth(\Delta R), \quad E < V_0. \tag{180}$$

Similarly, for $E > V_0$:

$$L_0 = kR \cot(kR + \delta_0) = \Delta' R \cot(\Delta' R), \quad E > V_0. \tag{181}$$

Given $V_0$ and $R$, we can thus solve these equations for $\delta_0(E)$.

Noting our earlier result:

$$1 + \alpha_0 = e^{2i\delta_0} = e^{-2ikR} \frac{1 + ikR/L_0}{1 - ikR/L_0}, \tag{182}$$

we have, for $E < V_0$:

$$e^{2i\delta_0} = e^{-2ikR} \frac{1 + i(k/\Delta)\tanh \Delta R}{1 - i(k/\Delta)\tanh \Delta R}, \quad E < V_0. \tag{183}$$

This is of the form $e^{-2ikR}\left(\frac{1+ia}{1-ia}\right) = e^{-2ikR}e^{i\theta}$, where $\theta = \tan^{-1}[2a/(1-a^2)]$. Therefore, we may write:

$$\delta_0 = -kR + \frac{1}{2}\tan^{-1}\left[\frac{2\frac{k}{\Delta}\tanh \Delta R}{1 - \left(\frac{k}{\Delta}\right)^2 \tanh^2 \Delta R}\right], \quad E < V_0, \tag{184}$$

$$\delta_0 = -kR + \frac{1}{2}\tan^{-1}\left[\frac{2\frac{k}{\Delta'}\tan \Delta' R}{1 - \left(\frac{k}{\Delta'}\right)^2 \tan^2 \Delta' R}\right], \quad E > V_0. \tag{185}$$

Consider the low energy limit, $k \to 0$ ($\Delta \to k_0$), $E < V_0$:

$$\delta_0 \to -kR + \frac{1}{2}\tan^{-1}\left[\frac{2\frac{k}{k_0}\tanh \Delta R}{1 - \left(\frac{k}{k_0}\right)^2 \tanh^2 \Delta R}\right] \tag{186}$$

$$= -kR + \frac{k}{k_0}\tanh k_0 R + O\left[\left(\frac{k}{k_0}\right)^2\right]. \tag{187}$$

Alternatively, we may consider the "hard sphere" limit, $k_0 \to \infty$ (*i.e.*, $V_0 \to \infty$). This is nearly the same limit, for any fixed energy. To lowest order in $k/k_0$,

$$\delta_0 = -kR + O(k/k_0). \tag{188}$$

Thus, for the hard sphere the lowest partial wave phase shift depends linearly on $k$, as shown in Fig. 6.

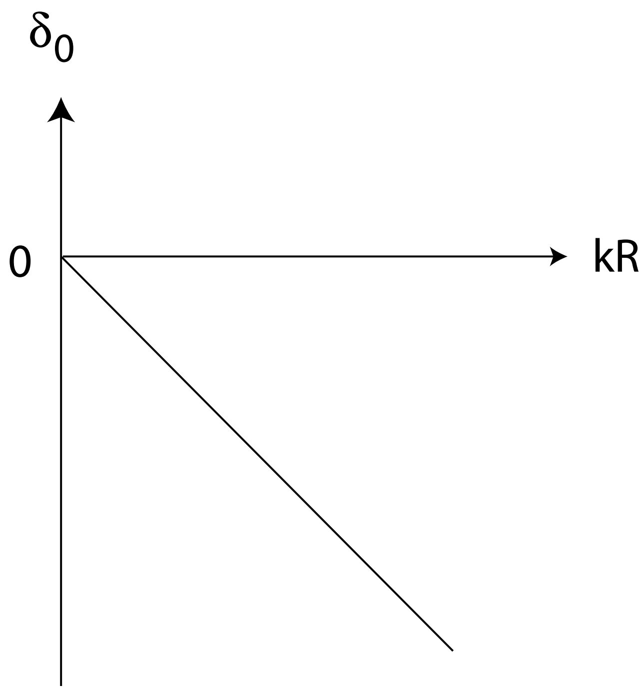


Figure 6: The S wave phase shift for the hard sphere potential.

Consider now the total cross section in the $\ell = 0$ channel:

$$\sigma_0 = \frac{4\pi}{k^2} \sin^2 \delta_0. \tag{189}$$

In the low energy limit, this is

$$\sigma_0 \to \frac{4\pi}{k^2} \sin^2 \left( -kR + \frac{k}{k_0} \tanh k_0 R \right) \tag{190}$$

$$\to \frac{4\pi}{k^2} \left( -kR + \frac{k}{k_0} \tanh k_0 R \right)^2 \tag{191}$$

$$= \quad 4\pi R^2 \left(1 - \frac{\tanh k_0 R}{k_0 R}\right)^2. \tag{192}$$

In the hard sphere limit ($k_0 \to \infty$), this becomes

$$\sigma_0 = 4\pi R^2. \tag{193}$$

In the low energy limit, this partial wave dominates, so this is the total cross section as well. This result is larger than the result we might expect purely geometrically, *i.e.*, for the scattering of a well-localized wave packet. The difference must be attributed to the wave nature of our probe, and the phenomenon of diffraction.

We note in passing that when writing

$$\sigma_0 = 4\pi a_0^2, \quad a_0 = R\left(1 - \frac{\tanh k_0 R}{k_0 R}\right), \tag{194}$$

$a_0$ is referred to as the "scattering length" of the potential.

Now let us consider the case of finite $V_0$ (a "soft sphere"?). For example, in the high energy limit, $k \gg k_0$, and $\Delta' \to k$, hence:

$$\delta_0 \quad \to \quad -kR + \frac{1}{2}\tan^{-1}\frac{2\tan kR}{1-\tan^2 kR} \tag{195}$$

$$= \quad 0. \tag{196}$$

This should be the expected result in the high energy limit, since the effect of a finite fixed potential becomes negligible as the energy increases.

Thus, at very low energies and at very high energies, the phase shift approaches zero. In between, it must reach some maximum (negative) value. Fig. 7 shows the $S$ wave phase shift for scattering on the soft sphere potential, and Fig. 8 shows the $S$ wave cross section for scattering on the soft sphere potential.

# 9 The Born Expansion, Born Approximation

We have mentioned the Born expansion in our note on resolvents and Green's functions, and the Born approximation in our note on approximate methods. We now revisit it explicitly in the context of scattering. We start by developing the approach in general, and then apply it to the example of the preceding section.

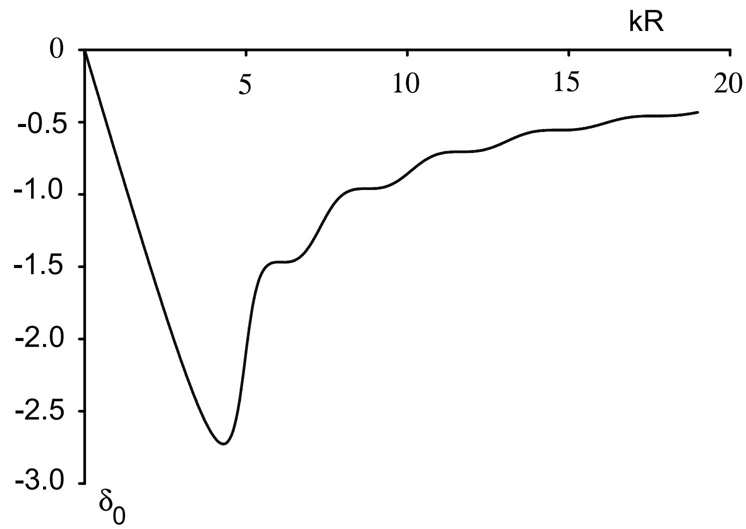


Figure 7: The $S$ wave phase shift for the soft sphere potential, with $R = 1$, $k_0 R = 4$.

In the resolvent note, we saw the Born expansion as the Liouville-Neumann expansion of a perturbed Green's function. We will see this again, except that it is convenient (eliminates a minus sign) here to *redefine the Green's function with the opposite sign from the resolvent note*. Thus, we here take

$$G(z) = -\frac{1}{H - z}. \tag{197}$$

The Schrödinger equation we wish to solve is of the form:

$$\left(-\frac{\nabla^2}{2m} + V\right)\psi = E\psi. \tag{198}$$

It is convenient to let $E = k^2/2m$ and $V = U/2m$, so that this equation is of the form:

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{x}) = U(\mathbf{x})\psi(\mathbf{x}). \tag{199}$$

We already found the Green's function for the free particle wave equation (Helmholtz equation), $(\nabla^2 + k^2)\psi = 0$ in the resolvent note:

$$G(\mathbf{x}, \mathbf{y}; k) = -\frac{e^{\pm ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x} - \mathbf{y}|}. \tag{200}$$

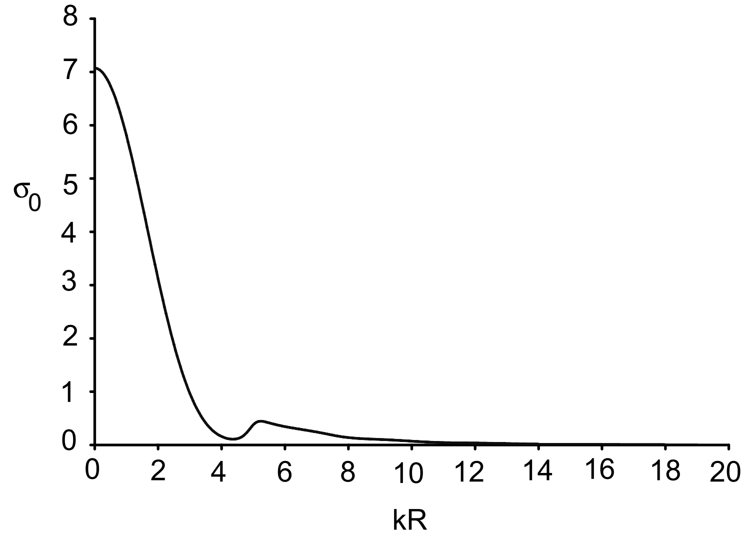


Figure 8: The $S$ wave cross section for the soft sphere potential, with $R = 1$, $k_0 R = 4$.

The solution to the desired Schrödinger equation is then

$$\psi(\mathbf{x}) = \psi_0(\mathbf{x}) + \int_{(\infty)} d^3(\mathbf{x}')G(\mathbf{x}, \mathbf{x}'; k)U(\mathbf{x}'\psi(\mathbf{x}'), \tag{201}$$

where $\psi_0(\mathbf{x})$ is a solution to the Helmholtz equation. The reader is encouraged to verify that $\psi$ is indeed a solution, by substituting into the Schrödinger equation.[4]

Now we may apply this to the scattering of an incident plane wave, $\psi_0 = e^{i\mathbf{k}\cdot\mathbf{x}}$, using the "outgoing" Green's function,

$$G_+(\mathbf{x}, \mathbf{y}; k) = -\frac{e^{+ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}, \tag{202}$$

to obtain the scattered wave:

$$\begin{aligned} \psi_S(\mathbf{x}) &\equiv \psi(\mathbf{x}) - \psi_0(\mathbf{x}) & (203) \\ &= \int_{(\infty)} d^3(\mathbf{x}')G_+(\mathbf{x}, \mathbf{x}'; k)U(\mathbf{x}'\psi(\mathbf{x}'). & (204) \end{aligned}$$

[4]Recall that $(\nabla_x^2 + k^2)G(\mathbf{x}, \mathbf{x}'; k) = \delta^{(3)}(\mathbf{x} - \mathbf{x}')$.

Since

$$\psi_S(\mathbf{x}) = f(k;\theta)\frac{e^{ikr}}{r}, \tag{205}$$

where $\theta$ is the polar angle to the observation point, hence equal to the polar angle of the scattered $\mathbf{k}$-vector, we can extract the scattering amplitude $f(k;\theta)$ by considering the asymptotic situation, where the source "size" (non-negligible region of $U(\mathbf{x})$) is small:

$$G_+(\mathbf{x},\mathbf{x}';k) \to_{|\mathbf{x}|\to\infty} -\frac{1}{4\pi}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}e^{-i\mathbf{k}'\cdot\mathbf{x}'}. \tag{206}$$

In obtaining this expression, we have held $\mathbf{x}'$ finite, and used

$$|\mathbf{x}-\mathbf{x}'| \approx |\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{x}'}{\mathbf{x}}. \tag{207}$$

Thus,

$$f(k;\theta) = -\frac{1}{4\pi}\int_{(\infty)} d^3(\mathbf{x}')e^{-i\mathbf{k}'\cdot\mathbf{x}'}U(\mathbf{x}')\psi(\mathbf{x}'). \tag{208}$$

Let us return to the general integral equation for the wave function:

$$\psi(\mathbf{x}) = \psi_0(\mathbf{x}) + \int_{(\infty)} d^3(\mathbf{x}')G(\mathbf{x},\mathbf{x}';k)U(\mathbf{x}')\psi(\mathbf{x}'). \tag{209}$$

We plug this expression into the integrand, obtaining:

$$\begin{aligned}\psi(\mathbf{x}) &= \psi_0(\mathbf{x}) + \int_{(\infty)} d^3(\mathbf{x})G(\mathbf{x},\mathbf{x}';k)U(\mathbf{x}')\psi_0(\mathbf{x}') \qquad (210)\\ &\quad + \int_{(\infty)} d^3(\mathbf{x}')d^3(\mathbf{x}'')G(\mathbf{x},\mathbf{x}';k)U(\mathbf{x}')G(\mathbf{x}',\mathbf{x}'';k)U(\mathbf{x}'')\psi(\mathbf{x}'').\end{aligned}$$

Iteration gives the Neumann series:

$$\begin{aligned}\psi(\mathbf{x}) &= \psi_0(\mathbf{x}) + \sum_{n=1}^{\infty}\int_{(\infty)} d^3(\mathbf{x}')d^3(\mathbf{x}^{n\prime})G(\mathbf{x},\mathbf{x}';k)U(\mathbf{x}') \qquad (211)\\ &\quad G(\mathbf{x}',\mathbf{x}'';k)U(\mathbf{x}'')\ldots G(\mathbf{x}^{n-1\prime},\mathbf{x}^{n\prime};k)U(\mathbf{x}^{n\prime})\psi_0(\mathbf{x}^{n\prime}). \qquad (212)\end{aligned}$$

Typically, we use this by substituting a plane wave for $\psi_0(\mathbf{x})$. If the potential is "weak", we expect this series to converge quickly. This expansion is sometimes called the *Born expansion*. To first order in $U(\mathbf{x})$ we have:

$$\psi(\mathbf{x}) = \psi_0(\mathbf{x}) + \int_{(\infty)} d^3(\mathbf{x}')G(\mathbf{x},\mathbf{x}';k)U(\mathbf{x}')\psi_0(\mathbf{x}') + O(U^2). \tag{213}$$

With $\psi_0(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}$, and

$$G(\mathbf{x}, \mathbf{x}'; k) \sim -\frac{1}{4\pi}\frac{e^{ikr}}{r}e^{-i\mathbf{k}'\cdot\mathbf{x}'}, \tag{214}$$

we find:

$$\psi(\mathbf{x}) \sim e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{1}{4\pi}\frac{e^{ikr}}{r}\int_{(\infty)} d^3(\mathbf{x}')e^{-i\mathbf{k}'\cdot\mathbf{x}'}U(\mathbf{x}')e^{i\mathbf{k}\cdot\mathbf{x}'}. \tag{215}$$

The interpretation of $\mathbf{k}$ is as the momentum vector of the incident plane wave, and of $\mathbf{k}'$ is as the momentum vector of the scattered wave. Hence,

$$\psi_S(\mathbf{x}) \approx -\frac{1}{4\pi}\frac{e^{ikr}}{r}\int_{(\infty)} d^3(\mathbf{x}')e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}U(\mathbf{x}'), \tag{216}$$

or

$$f(k; \Omega') \approx -\frac{1}{4\pi}\int_{(\infty)} d^3(\mathbf{x}')e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}U(\mathbf{x}'), \tag{217}$$

where $\Omega'$ is the direction of the scattered wave, *i.e.*, the direction of $\mathbf{k}'$. This result is known as the Born approximation.

With this result, we have the differential cross section:

$$\frac{d\sigma}{d\Omega'} = |f(k; \Omega')|^2 \tag{218}$$

$$= \frac{1}{16\pi^2}\left|\int_{(\infty)} d^3(\mathbf{x}')e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}U(\mathbf{x}')\right|^2 \tag{219}$$

$$= \frac{m^2}{4\pi^2}\left|\hat{V}(\mathbf{k}' - \mathbf{k})\right|^2, \tag{220}$$

where we have used $U(\mathbf{x}) = 2mV(\mathbf{x})$, and $\hat{V}$ is the Fourier transform of $V$. Recall that we already derived this result in our discussion of time-dependent perturbation theory (Approximate Methods course note).

If $U(\mathbf{x}) = U(r)$ only, *i.e.*, we have a spherically symmetric scattering potential, we may incorporate this into our result. Let $\theta$ be the "scattering angle", *i.e.*, the angle between $\mathbf{k}$ and $\mathbf{k}'$. Then, noting that $|\mathbf{k}| = |\mathbf{k}'| = k$, we have $|\mathbf{k} - \mathbf{k}'| = 2k\sin\frac{\theta}{2}$. Since $U(\mathbf{x}) = U(r)$, $f(k; \Omega') = f(k, \theta)$ only, and

$$f(k; \theta) = -\frac{1}{4\pi}\int_0^\infty r'^2 dr' U(r')\int_{4\pi} d\cos\theta' d\phi' e^{i2k(\sin\frac{\theta}{2})r'\cos\theta'}, \tag{221}$$

where $\theta'$ is the angle between $\mathbf{x}'$ and $\mathbf{k}-\mathbf{k}'$. Now let $K = |\mathbf{k}-\mathbf{k}'| = 2k\sin\frac{\theta}{2}$. We can do the integrals over angles, to obtain:

$$f(k;\theta) = -\int_0^\infty r'^2 dr' U(r') \frac{\sin Kr'}{Kr'}. \tag{222}$$

This is the Born approximation for a spherically symmetric potential.

## 9.1 Born Approximation for Phase Shifts

Now let us consider what the Born approximation yields for the phase shifts in a partial wave expansion. Start with our earlier (exact asymptotic) result:

$$\psi(\mathbf{x}) = \psi_0(\mathbf{x}) + \psi_S(\mathbf{x}) \tag{223}$$

$$= \psi_0(\mathbf{x}) + \frac{e^{ikr}}{r} f(k;\theta), \tag{224}$$

where

$$f(k;\theta) = -\frac{1}{4\pi}\int_{(\infty)} d^3(\mathbf{x}') e^{-i\mathbf{k}'\cdot\mathbf{x}'} U(\mathbf{x}')\psi(\mathbf{x}'). \tag{225}$$

The expansion of $\psi(\mathbf{x})$ in partial waves is

$$\psi(\mathbf{x}) = \sum_{\ell=0}^\infty i^\ell (2\ell+1) R_\ell(r) P_\ell(\cos\theta). \tag{226}$$

We insert this into the expression for $f$:

$$f(k;\theta) = -\frac{1}{4\pi}\sum_{\ell=0}^\infty i^\ell(2\ell+1)\int_0^\infty r'^2 dr' R_\ell(r')\int_{-1}^1 d\cos\theta' \int_0^{2\pi} d\phi' e^{-i\mathbf{k}'\cdot\mathbf{x}'} U(\mathbf{x}') P_\ell(\cos\theta'). \tag{227}$$

If, again, we have a spherical potential, $U(\mathbf{x}) = U(r)$ only, then we can perform the angular integrals. Note that $\theta$ is the scattering angle, *i.e.*, the angle between $\mathbf{k}$ and $\mathbf{k}'$. If $\mathbf{k}$ is along the $z$ axis, then $\theta$ is the polar angle of $\mathbf{k}'$. Thus, $\mathbf{k}'$ is not necessarily along the $z$-axis, and we have:

$$\mathbf{k}'\cdot\mathbf{x}' = kr' \left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi_{k'} - \phi_{x'})\right]. \tag{228}$$

Now consider our partial wave expansion of a plane wave:

$$e^{i\mathbf{k}\cdot\mathbf{x}} = e^{ikr\cos\theta_{kx}} = \sum_{\ell=0}^\infty i^\ell(2\ell+1) j_\ell(kr) P_\ell(\cos\theta_{kx}) \tag{229}$$

$$= \sum_{\ell=0}^\infty i^\ell \sqrt{4\pi(2\ell+1)}\, j_\ell(kr) Y_{\ell 0}(\theta_{kx}, \phi_{kx}), \tag{230}$$

where $\theta_{kx}$ is the angle between $\mathbf{k}$ and $\mathbf{x}$, and we have used the identity

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell 0}(\theta, \phi). \tag{231}$$

We may re-express this in terms of the polar angles $(\theta_x, \phi_x)$ of $\mathbf{x}$ and $(\theta_k, \phi_k)$ of $\mathbf{k}$ using the addition theorem for spherical harmonics:

$$\sqrt{4\pi(2\ell+1)} Y_{\ell 0}(\theta_{kx}, \phi_{kx}) = \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta_k, \phi_k) Y_{\ell m}(\theta_x, \phi_x), \tag{232}$$

where

$$\cos\theta_{kx} = \cos\theta_k \cos\theta_x + \sin\theta_k \sin\theta_x \cos(\phi_k - \phi_x). \tag{233}$$

Thus,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi \sum_{\ell=0}^{\infty} i^\ell j_\ell(kr) \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta_k, \phi_k) Y_{\ell m}(\theta_x, \phi_x), \tag{234}$$

and hence,

$$f(k;\theta) = -\frac{1}{4\pi} \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) \int_0^\infty r'^2 dr' R_\ell(r') \tag{235}$$

$$4\pi \sum_{\ell'=0}^{\infty} i^{\ell'} j^*_{\ell'}(k'r') U(r') \tag{236}$$

$$\int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\phi' \sum_{m=-\ell}^{\ell} Y_{\ell' m}(\theta, \phi_{k'}) Y^*_{\ell' m}(\theta', \phi') P_\ell(\cos\theta') \tag{237}$$

We may simplify this in several ways:

- $j_\ell(x)$ is real if $x$ is real.
- We may relabel integration variable $r'$ as $r$. Also, $k'$ is equal to $k$.
- The only dependence on $\phi'$ is in $Y^*_{\ell' m}(\theta', \phi')$, hence

$$\int_0^{2\pi} d\phi' Y^*_{\ell' m}(\theta', \phi') = 2\pi \delta_{m0} Y^*_{\ell' 0}(\theta', 0) \tag{238}$$

$$= 2\pi \delta_{m0} \sqrt{\frac{2\ell'}{4\pi}} P_{\ell'}(\cos\theta'). \tag{239}$$

- The integral over $\cos\theta'$ may then be accomplished with:

$$\int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}. \tag{240}$$

- Finally, we may use

$$Y_{\ell 0}(\theta, 0) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta). \tag{241}$$

Including all of these points leads to:

$$f(k;\theta) = -\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)\int_0^\infty r^2 dr R_\ell(r) j_\ell(kr) U(r). \tag{242}$$

So far, we have made no approximation – this result is "exact" as long as the potential falls off rapidly enough as $r \to \infty$. However, it depends on knowing $R_\ell(r)$, and this is where we shall now make an approximation. Note first, that by comparison with the expansion in phase shifts:

$$f(k;\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left(e^{2i\delta_\ell} - 1\right) P_\ell(\cos\theta), \tag{243}$$

we obtain:

$$\frac{1}{2i}\left(e^{2i\delta_\ell} - 1\right) = e^{i\delta_\ell}\sin\delta_\ell = -k\int_0^\infty r^2 dr R_\ell(r) j_\ell(kr) U(r). \tag{244}$$

The Born approximation consists in approximating the wave function by its value for no potential, which should be a good approximation as long as the potential has a small effect on it. Here, this means we replace $R_\ell(r)$ with $j_\ell(kr)$, valid as long as $R_\ell(r) \approx j_\ell(kr)$:

$$e^{i\delta_\ell}\sin\delta_\ell \approx -k\int_0^\infty r^2 dr \left[j_\ell(kr)\right]^2 U(r). \tag{245}$$

Note that the integral is real in this approximation. In this approximation, $\delta_\ell$ is small, hence we have the Born approximation for the phase shifts:

$$\delta_\ell \approx -k\int_0^\infty r^2 dr \left[j_\ell(kr)\right]^2 U(r). \tag{246}$$

## 9.2 Born Approximation and Example of the "Soft Sphere" Potential

As an example, let us return to our earlier example of the "soft sphere":

$$V(r) = \begin{cases} V_0, & r < R, \\ 0, & r > R. \end{cases} \tag{247}$$

Now $U(r) = 2mV(r)$, and let $k_0^2 = 2mV_0$. Then,

$$\delta_\ell \approx -kk_0^2 \int_0^R r^2 dr \, [j_\ell(kr)]^2 \tag{248}$$

$$= -\left(\frac{k_0}{k}\right)^2 \int_0^{kR} [xj_\ell(x)]^2 \, dx. \tag{249}$$

For instance, consider $\delta_0$, with $j_0(x) = \frac{\sin x}{x}$:

$$\delta_0 \approx -\left(\frac{k_0}{k}\right)^2 \int_0^{kR} \sin^2 x dx \tag{250}$$

$$= -\left(\frac{k_0}{k}\right)^2 \frac{1}{4}(2kR - \sin 2kR). \tag{251}$$

Note that the Born approximation is a "high energy" approximation – there must be little effect on the scattering wave. Consider what happens at low energy, $kR \ll 1$:

$$\delta_0 = -\left(\frac{k_0}{k}\right)^2 \frac{1}{4}(2kR - \sin 2kR) \approx -kR\frac{(k_0R)^3}{3}. \tag{252}$$

This may be compared with our earlier low energy result:

$$\delta_0 \approx -kR\left(1 - \frac{\tanh k_0R}{k_0R}\right). \tag{253}$$

Other than the linear dependence, they are not very similar. Fig. 9 illustrates this. At low $kR$, the agreement with the exact calculation is poor, as the energy increases, the agreement improves.

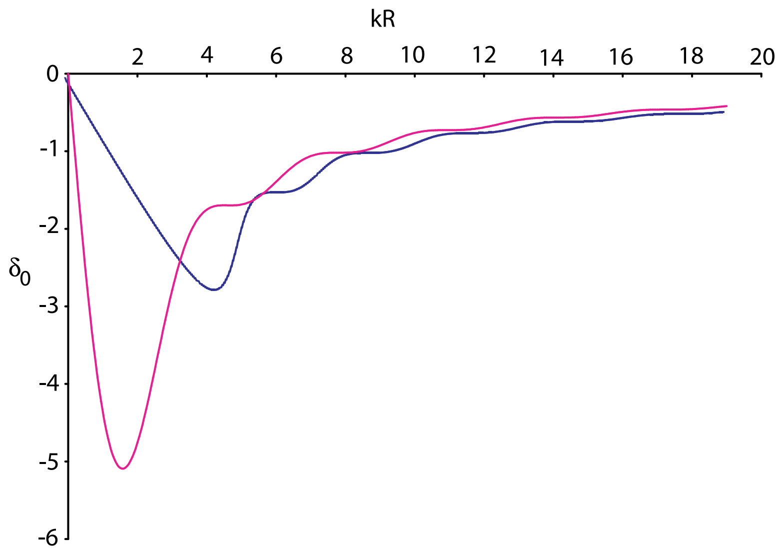


Figure 9: The S wave phase shift for the soft sphere potential, with $R = 1$, $k_0R = 4$. The curve which goes further negative is the Born approximation; the other curve is the exact result.

# 10 Angular Distributions

If the laws of physics are rotationally invariant, we have angular momentum conservation. As a consequence, the angular distribution of particles in a scattering process is constrained. Here, we consider how this may be applied in the case of a scattering reaction of the form:

$$a + b \to c + d. \tag{254}$$

These particles may carry spins – let $j_a$ denote the spin of particle $a$, etc. It is convenient to work in the center-of-mass frame, and to pick particle $a$ to be incoming along the $z$ axis. If a different frame is desired, then it may be reached from the center-of-mass frame with a Gallilean or Lorentz transformation, whichever is appropriate.

It is furthermore convenient to work in a helicity basis for the angular momentum states. Along the helicity axis, the component of orbital angular momentum of a particle is zero. We'll typically label the helicity of a particle with the symbol $\lambda$. For example, the helicity of particle $a$ is denoted $\lambda_a$.

Let us also work with a plane wave basis for our particles. Physical states may be constructed as a superposition of such states. We'll omit the magnitude of momentum in our labelling, and concentrate on the angles (polar angles denoted by $\theta$, and azimuth angles denoted by $\phi$). We'll also omit the spins of the scattering particles in the labels, to keep things compact. Thus, we label the initial state according to:

$$|i\rangle = |\theta_a = 0, \phi_a, \lambda_a, \lambda_b\rangle, \tag{255}$$

and the final state by,

$$|f\rangle = |\theta_c, \phi_c, \lambda_c, \lambda_d\rangle, \tag{256}$$

Note that $\theta_d = \pi - \theta_c$ and $\phi_d = \pi + \phi_c$ in the center-of-mass. This basis may be called the "plane wave helicity basis".

We may write the transition amplitude from the initial to final state in the form:

$$\langle f|T|i\rangle \propto \langle \theta_c, \phi_c, \lambda_c, \lambda_d|T|0, \phi_a, \lambda_a, \lambda_b\rangle, \tag{257}$$

and the scattering cross section is:

$$d\sigma_{\lambda_a\lambda_b\lambda_c\lambda_d}(\theta \equiv \theta_c, \phi \equiv \phi_c) \propto |\langle \theta, \phi, \lambda_c, \lambda_d|T|0, \phi_a, \lambda_a, \lambda_b\rangle|^2. \tag{258}$$

This gives the scattering cross section for the specified helicity states.

If we do not measure the final helicities, then we must sum over all possible $\lambda_c$ and $\lambda_d$. Similarly, if the initial beams ($a$ and $b$) are unpolarized, we must average over all possible $\lambda_a$ and $\lambda_b$. Thus, to get the scattering cross section in such a case, we must evaluate

$$\frac{1}{2j_a+1}\sum_{\lambda_a}\frac{1}{2j_b+1}\sum_{\lambda_b}\sum_{\lambda_c}\sum_{\lambda_d}|\langle \theta, \phi, \lambda_c, \lambda_d|T|0, \phi_a, \lambda_a, \lambda_b\rangle|^2. \tag{259}$$

If $a$(or $b$) is massless and not spinless, then the $1/(2j_a+1)$ factor must be replaced by $1/2$.

The scattering process may go via intermediate states of definite angular momentum, for example, scattering through an intermediate resonance. For example. in $e^+e^-$ scattering just below $\mu^+\mu^-$ threshold, we may expect a resonance due to the presence of "muonium" bound states. Such states have definite angular momenta. In particular, the resonance for scattering through the lowest $^3S_1$ bound state of muonium might be expected to be a large contribution in the scattering amplitude. Thus, we are led to describing the transition amplitude most naturally in a basis which specifies the total angular momentum state. We may label a basis state as:

$$|j, m, \lambda_a, \lambda_b\rangle, \quad \text{or} \quad |j, m, \lambda_c, \lambda_d\rangle, \tag{260}$$

where $j$ specifies the total angular momentum, and $m$ is its component along a chosen quantization axis. This basis is sometimes called the "spherical helicity basis".

For a given pair of initial and final helicity states, we are thus interested in computing matrix elements of the form:

$$\langle f|T|i\rangle \propto \sum_{j,m}\langle\theta_c, \phi_c, \lambda_c, \lambda_d|j, m, \lambda_c, \lambda_d\rangle\langle j, m, \lambda_c, \lambda_d|T \sum_{j',m'}|j', m', \lambda_a, \lambda_b\rangle\langle j', m', \lambda_a, \lambda_b|0, \phi_a, \lambda_a, \lambda_b\rangle. \tag{261}$$

The terms of the form $\langle\theta, \phi, \lambda_1, \lambda_2|j, m, \lambda'_1, \lambda'_2\rangle$ are just scalar products of vectors expressed in different bases. We thus need to learn how to do the basis transformation.

We may write the basis transformation in the form:

$$|\theta, \phi, \lambda_1, \lambda_2\rangle = \sum_{j,m,\lambda'_1,\lambda'_2}|j, m, \lambda'_1, \lambda'_2\rangle\langle j, m, \lambda'_1, \lambda'_2|\theta, \phi, \lambda_1, \lambda_2\rangle \tag{262}$$

$$= \sum_{j,m}|j, m, \lambda_1, \lambda_2\rangle\langle j, m, \lambda_1, \lambda_2|\theta, \phi, \lambda_1, \lambda_2\rangle, \tag{263}$$

where we have used the fact that states with different helicities are orthogonal:

$$\langle j, m, \lambda'_1, \lambda'_2|\theta, \phi, \lambda_1, \lambda_2\rangle \propto \delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2}. \tag{264}$$

Consider now the state

$$|\theta = 0, \phi, \lambda_1, \lambda_2\rangle = \sum_{j,m}|j, m, \lambda_1, \lambda_2\rangle\langle j, m, \lambda_1, \lambda_2|\theta = 0, \phi, \lambda_1, \lambda_2\rangle. \tag{265}$$

For this state, the helicity axis is parallel(or antiparallel) to the quantization axis for the third component of angular momentum, and hence $\lambda_1 = m_1$, $\lambda_2 = -m_2$. Thus, $m = \lambda_1 - \lambda_2 \equiv \alpha$. The coefficient

$$\langle j, m, \lambda'_1, \lambda'_2|\theta, \phi, \lambda_1, \lambda_2\rangle = c_j\delta_{m\alpha}, \tag{266}$$

where $c_j$ is a constant depending only on $j$, to be determined (see exercises). The expansion of this state in the spherical helicity basis is thus:

$$|\theta = 0, \phi, \lambda_1, \lambda_2\rangle = \sum_{j,m} c_j|j, \alpha, \lambda_1, \lambda_2\rangle. \tag{267}$$

Let us now perform a rotation on our special state to an arbitrary state:

$$|\theta, \phi, \lambda_1, \lambda_2\rangle = R_3(\phi)R_2(\theta)R_3(-\phi)|\theta = 0, \phi, \lambda_1, \lambda_2\rangle. \tag{268}$$

This product of rotations on a state of angular momentum $j$ is given by the Euler angle parameterization of the $D^j$ rotation matirces:

$$R_3(\phi)R_2(\theta)R_3(-\phi) = D^j(\phi, \theta, -\phi). \tag{269}$$

The matrix elements for rotation $u$ are:

$$D^j_{m_1m_2}(u) = \langle j, m_1|D^j(u)|j, m_2\rangle. \tag{270}$$

Hence,

$$D^j(u)|j, \alpha, \lambda_1, \lambda_2\rangle = \sum_m |j, m, \lambda_1, \lambda_2\rangle\langle j, m, \lambda_1, \lambda_2|D^j(u)|j, \alpha, \lambda_1, \lambda_2\rangle, \tag{271}$$

and

$$ket\theta, \phi, \lambda_1, \lambda_2 = \sum_{j,m} |j, m, \lambda_1, \lambda_2\rangle D^j_{m\alpha}(\phi, \theta, -\phi). \tag{272}$$

If we anticipate the result of the exercises for $c_j$, namely $c_j = \sqrt{2j + 1/4\pi}$, we have the desired result:

$$\langle\theta, \phi, \lambda_1, \lambda_2|j, m, \lambda_1', \lambda_2'\rangle = \sqrt{\frac{2j+1}{4\pi}} D^{j*}_{m\alpha}(\phi, \theta, -\phi)\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}, \tag{273}$$

where $\alpha \equiv \lambda_1 - \lambda_2$. Putting this into our expression for the transition amplitude, we obtain (defining $\alpha_f \equiv \lambda_c - \lambda_d$ and $\alpha_i \equiv \lambda_a - \lambda_b$):

$$\langle f|T|i\rangle \propto \sum_{j,m}\sum_{j',m'} \sqrt{\frac{2j+1}{4\pi}}\sqrt{\frac{2j'+1}{4\pi}} D^{j*}_{m\alpha_f}(\phi, \theta, -\phi)D^{j'}_{m'\alpha_i}(\phi_a, 0, -\phi_a)\langle j, m, \lambda_c, \lambda_d|T|j', m', \lambda_a, \lambda_b\rangle. \tag{274}$$

Suppose the interaction is one which conserves angular momentum. In this case,

$$\langle j, m, \lambda_c, \lambda_d|T|j', m', \lambda_a, \lambda_b\rangle = \delta_{jj'}\delta mm'\langle j, m, \lambda_c, \lambda_d|T|j, m, \lambda_a, \lambda_b\rangle, \tag{275}$$

and further $m = \lambda_a - \lambda_b \equiv \alpha_i$. Thus, the non-zero transition matrix elements are numbers of the form $T^j_{\lambda_a\lambda_b\lambda_c\lambda_d}$, which may be called "helicity amplitudes". We have:

$$\langle f|T|i\rangle \propto \sum_j \frac{2j+1}{4\pi} T^j_{\lambda_a\lambda_b\lambda_c\lambda_d} D^{j*}_{\alpha_i\alpha_f}(\phi, \theta, -\phi)D^j_{\alpha_i\alpha_i}(\phi_a, 0, -\phi_a). \tag{276}$$

The "big-$D$" functions give us the angular distribution for each intermediate $j$ value and helicity state. Using

$$D^j_{m_1 m_2}(\phi, \theta, -\phi) = e^{-i(m_1 - m_2)\phi} d^j_{m_1 m_2}(\theta), \tag{277}$$

and $d^j_{mm}(0) = 1$, we have

$$\langle f|T|i\rangle \propto \sum_j \frac{2j+1}{4\pi} T^j_{\lambda_a\lambda_b\lambda_c\lambda_d} e^{i(\alpha_i - \alpha_f)\phi} d^j_{\alpha_i\alpha_f}(\theta). \tag{278}$$

Note that the azimuth dependence is a phase, depending only on the helicities.

Squaring, we obtain the scattering angular distribution:

$$\frac{d\sigma_{\lambda_a\lambda_b\lambda_c\lambda_d}}{d\Omega}(\theta, \phi) \propto \left| \sum_j (2j+1) T^j_{\lambda_a\lambda_b\lambda_c\lambda_d} d^j_{\alpha_i\alpha_f}(\theta) \right|^2, \tag{279}$$

where $\alpha_f \equiv \lambda_c - \lambda_d$ and $\alpha_i \equiv \lambda_a - \lambda_b$.

# 11 Exercises

1. Show that the total cross section we computed in the partial wave expansion,

$$\sigma_T(p) = \frac{4\pi}{p^2} \sum_{j=0}^{\infty} (2j+1) \sin^2 \delta_j(p), \tag{280}$$

is in agreement with the optical theorem.

2. We have discussed the "central force problem". Consider a particle of mass $m$ under the influence of the following potential:

$$V(r) = \begin{cases} V_0, & 0 \le r \le a \\ 0, & a < r, \end{cases} \tag{281}$$

where $V_0$ is a constant.

(a) Write down the Schrödinger equation for the wave function $\psi(\mathbf{x})$. Consider solutions which are simultaneous eigenvectors of $H$, $\mathbf{L}^2$, and $L_z$. Solve the angular dependence, and reduce the remaining problem to a problem in one variable. [You've done this already first quarter, so you may simply retrieve that result here.]

(b) Let $E$ be the eigenvalue of the Hamiltonian, $H$. Consider the case where $E > V_0$. Solve the Schrödinger equation for eigenstates $\psi(\mathbf{x})$. It will probably be convenient to use the quantity $k = \sqrt{2m(E - V_0)}$. Consider the limit as $r \to \infty$ for your solutions, and give an interpretation in terms of spherical waves.

(c) Repeat the solution for the case where $E < V_0$. It will probably be convenient to use the quantity $K = \sqrt{2m(V_0 - E)}$. Again, consider the limit as $r \to \infty$ and give an interpretation, contrasting with the previous case.

Hint: You will probably benefit by thinking about solutions in the form of spherical Bessel/Neumann functions, and/or spherical Hankel functions.

3. When we calculated the density of states for a free particle, we used a "box" of length $L$ (in one dimension), and imposed periodic boundary conditions to ensure no net flux of particles into or out of the box. We have in mind, of course, that we can eventually let $L \to \infty$, and are really interested in quantities per unit length (or volume). Let us justify more carefully the use of periodic boundary conditions, *i.e.*, we wish to convince ourselves that the intuitive rationale given above is correct. To do this, consider a free particle in a one-dimensional "box" from $-L/2$ to $L/2$. Remembering that the Hilbert space of allowed states is a linear space, show that the periodic boundary condition:

$$\psi(-L/2) = \psi(L/2),$$

is required for acceptable wave functions. "Acceptable" here means that the probability to find a particle in the box must be constant.

4. In our discussion of scattering theory, we supposed we had a beam of particles from some ensemble of wave packets, and obtained an "effective" (observed) differential cross-section:

$$\sigma_{\text{eff}}(\mathbf{u}) = \int_{\{\alpha\}} f(\alpha) d\alpha \int_{|\mathbf{x}| \leq R} d^2(\mathbf{x}) P(\mu; \propto; \mathbf{x})$$

This formula assumed that the beam particles were distributed uniformly in a disk of radius $R$ centered at the origin in the $\hat{e}_1 - \hat{e}_2$ plane, and that the distribution of the shape parameter was uncorrelated with position in this disk.

(a) Try to obtain an expression for $\sigma_{\text{eff}}(\mathbf{u})$ without making these assumptions.

(b) Using part (a), write down an expression for $\sigma_{\text{eff}}(\mathbf{u})$ appropriate to the case where the beam particles are distributed according to a Gaussian of standard deviation $\rho$ in radial distance from the origin (in the $\hat{e}_2 - \hat{e}_3$ plane), and where the wave packets are also drawn from a Gaussian distribution in the expectation value of the magnitude of the momentum. Let the standard deviation of this momentum distribution be $\alpha = \alpha(\mathbf{x})$, for beam position $\mathbf{x}$.

(c) For your generalized result of part (a), try to repeat our limiting case argument to obtain the "fundamental" cross section. Discuss.

5. Let us briefly consider the consequences of reflection invariance (parity conservation) for the scattering of a particle with spin $s$ on a spinless target. [We consider elastic scattering only here]. Thus, assume the interaction is reflection invariant:

   (a) How does the $S$ matrix transform under parity, *i.e.*, what is $P^{-1}SP$, where $P$ is the parity operator?

   (b) What is the condition on the helicity amplitudes $A^j_{\lambda\mu}(p_i)$ (corresponding to scattering with total angular momentum $j$) imposed by parity conservation?

   (c) What condition is imposed on the orbital angular momentum amplitudes $B^j_{\ell\ell'}(p_i)$? You may use "physical intuition" if you like, but it should be convincing. In any event, be sure your answer makes intuitive sense.

6. We consider the resonant scattering of light by an atom. In particular, let us consider sodium, with ${}^2P_{1/2} \leftrightarrow {}^2 S_{1/2}$ resonance at $\lambda = \lambda_0 = 5986\text{Å}$. Let $\sigma_{0T}$ be the total cross section at resonance, for a monochromatic light source (*i.e.*, $\sigma_{0T}$ is the "fundamental" cross section).

   (a) Ignoring spin, estimate $\sigma_{0T}$, first in terms of $\lambda_0/2\pi$, and then numerically in cm$^2$. Compare your answer with a typical atomic size.

(b) Suppose that we have a sodium lamp source with a line width governed by the mean life of the excited $^2P_{1/2}$ state (maybe not easy to get this piece of equipment!). The mean life of this state is about $10^{-8}$ second. Suppose that this light is incident on an absorption cell, containing sodium vapor and an inert (non-resonant) buffer gas. Let the temperature of the gas in the absorption cell be 200°C. Obtain an expression for the effective total cross section, $\sigma_{\text{eff}T}$ which an atom in the cell presents to the incident light. Again, make a numerical calculation in $\text{cm}^2$.

(c) Using your result above, find the number density of Na atoms (# of atoms/$\text{cm}^3$) which is required in the cell in order that intensity of the incident light is reduced by a factor of two in a distance of 1 cm. It should be noted (and your answer should be plausible here) that such a gas will be essentially completely transparent to light of other (non-resonant) wavelengths.

7. Consider scattering from the simple potential:

$$V(\mathbf{x}) = \begin{cases} V_0 & r = |\mathbf{x}| < R \\ 0 & r > R. \end{cases}$$

In the low energy limit, we might only look at $S$-wave $\ell = 0$ scattering. However, in the high energy limit, we expect scattering in other partial waves to become significant. For simplicity, let us here consider scattering on a hard sphere, $V_0 \to \infty$.

(a) For a hard sphere potential, calculate the total cross section in partial wave $\ell$. Give the exact result, *i.e.*, don't take the high energy limit yet. You may quote your answer in terms of the spherical Bessel functions.

(b) Find a simple expression for the phase shift $\delta_\ell$ in the high energy limit ($kR \gg \ell$). Keep terms up to O(1) in your result.

(c) Determine the total cross section (including all partial waves) in the high energy limit, $kR \to \infty$. [This is the only somewhat tricky part of this problem to calculate. One approach is as follows: Write down the total cross section in terms of your results for part (a). Then, for fixed $k$, consider which values of $\ell$ may be important in the sum. Neglect the other values of $\ell$, and make the high energy approximation to your part (a) result. Finally, evaluate the sum, either directly, or by turning it into an appropriate integral.]

8. Consider the graph in Fig. 10.

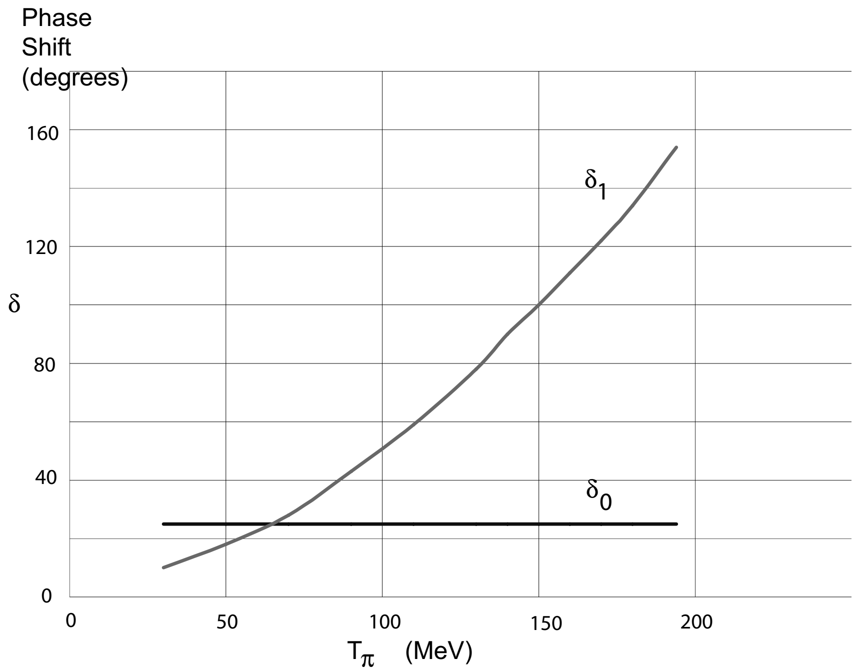


Figure 10: Made-up graph of phase shifts $\delta_0$ and $\delta_1$ for elastic $\pi^+p$ scattering (neglecting spin).

Assume that the other phase shifts are negligible (*e.g.*, "low energy" is reasonably accurate). The pion mass and energy here are sufficiently small that we can at least entertain the approximation of an infinitely heavy proton at rest – we'll assume this to be the case, in any event. Note that $T_\pi$ is the relativistic kinetic energy of the $\pi^+ : T_\pi = \sqrt{P_\pi^2 + m_\pi^2} - m_\pi$.

(a) Is the $\pi^+p$ force principally attractive or repulsive (as shown in this figure)?

(b) Plot the total cross section in mb (millibarns) as a function of energy, from $T_\pi$=40 to 200 MeV.

(c) Plot the angular distribution of the scattered $\pi^+$ at energies of 120, 140 and 160 MeV.

(d) What is the mean free path of 140 MeV pions in a liquid hydrogen target, with these "protons"?

9. We now start to consider the possibility of "inelastic scatting". For example, let us suppose there is a "multiplet" of $N$ non-identical particles, all of mass $m$. We consider scattering on a spherically symmetric center-of-force, with the properly that the interaction can change a particle from one number of the multiplet to another member. We may in this case express the scattering amplitude by $f_{\alpha\beta}(k; \cos\theta)$, with $\alpha, \beta = 1, \ldots, N$, corresponding to a unitary $S$-matrix on Hilbert space $L_2(R^3) \otimes V_N$:

$$S_{\alpha\beta}(\mathbf{k}_f; \mathbf{k}_i) = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{k}_f - \mathbf{k}_i) + \frac{i}{2\pi k_i}\delta(k_f - k_i) f_{\alpha\beta}(\mathbf{k}_f; \mathbf{k}_i)$$

$\beta$ is here to be interpreted as identifying the initial particle, and $\alpha$ the final particle. The generalization of our partial wave expansion to this situation is clearly:

$$f_{\alpha\beta}(k; \cos\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(A^{(\ell)}_{\alpha\beta} - \delta_{\alpha\beta})P_\ell(\cos\theta)$$

Where $A^{(\ell)}$ is an $N \times N$ unitary matrix:

$$A^{(\ell)} = \exp[2i\triangle_\ell(k)]$$

with $\triangle_\ell$ an $N \times N$ Hermitian "phase shift matrix" note that $f_{\alpha\alpha}$ is the elastic scattering amplitude for particle $\alpha$.

(a) Find expressions, in terms of $A^{(\ell)}(k)$, for the following total cross sections, for an incident particle $\alpha$: (integrated over angles)

i. $\sigma^{\text{el}}_{\alpha\text{TOT}}$, the total elastic cross section

ii. $\sigma^{\text{inel}}_{\alpha\text{TOT}}$, the total inelastic cross section (sometimes called the "reaction" cross section).

iii. $\sigma_{\alpha\text{TOT}}$, the total cross section.

(b) Try to give the generalization of the optical theorem for this scattering of particles in a multiplet.

10. In the previous problem you considered the scattering of particles in a multiplet. You determined the total elastic (sometimes called "scattering") cross section and the total inelastic ("reaction") cross sections in terms of the $A^{(\ell)}_{\alpha\beta}$ matrix in the partial wave expansion. Consider now the graph in Fig. 11.

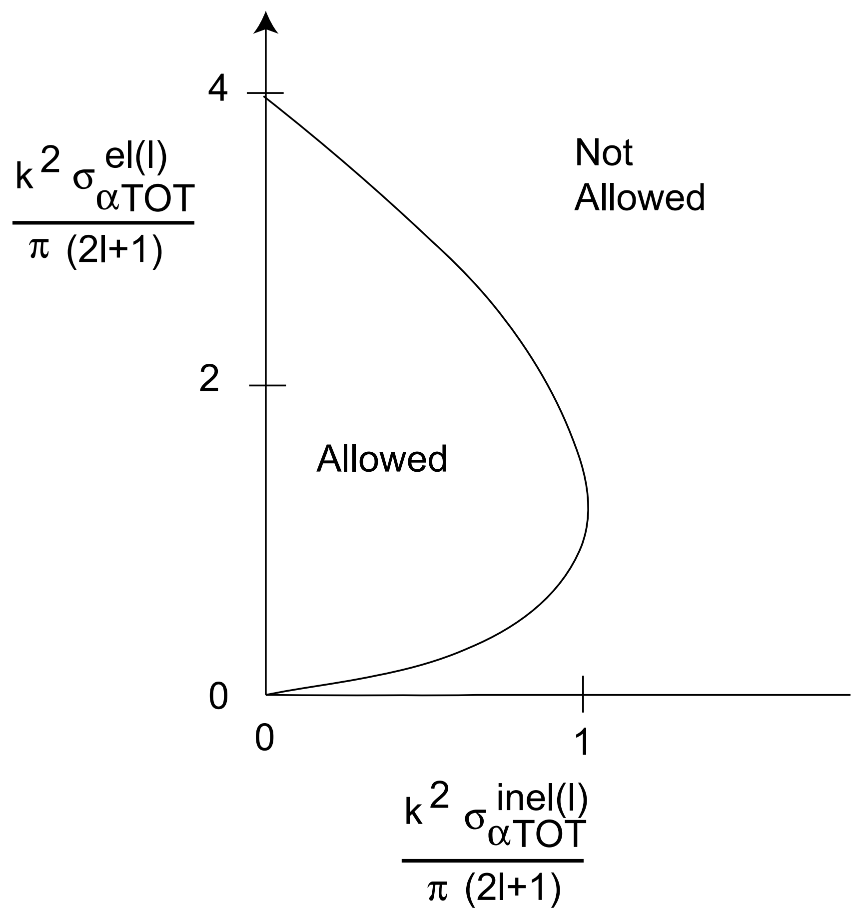


Figure 11: The allowed and forbidden regions for possible elastic and inelastic cross sections for the scattering of particles in a multiplet.

This graph purports to show the allowed and forbidden regions for the total elastic and inelastic cross sections in a given partial wave $\ell$. Derive the formula for the allowed region of this graph. Make sure to check the extreme points.

11. In the angular distribution section, we discussed the transformation between two different types of "helicity bases". In particular, we considered a system of two particles, with spins $j_1$ and $j_2$, in their CM frame.

    One basis is the "spherical helicity basis", with vectors of the form:

$$|j, m, \lambda_1, \lambda_2\rangle, \tag{282}$$

    where $j$ is the total angular momentum, $m$ is the total angular momentum projection along the 3-axis, and $\lambda_1, \lambda_2$ are the helicities of the two particles. We assumed a normalization of these basis vectors such that:

$$\langle j', m', \lambda_1', \lambda_2' | j, m, \lambda_1, \lambda_2\rangle = \delta_{jj'}\delta_{mm'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}. \tag{283}$$

    The other basis is the "plane-wave helicity basis", with vectors of the form:

$$|\theta, \phi, \lambda_1, \lambda_2\rangle, \tag{284}$$

    where $\theta$ and $\phi$ are the spherical polar angles of the direction of particle one. We did not specify a normalization for these basis vectors, but an obvious (and conventional) choice is:

$$\langle \theta', \phi', \lambda_1', \lambda_2' | \theta, \phi, \lambda_1, \lambda_2\rangle = \delta^{(2)}(\Omega' - \Omega)\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}, \tag{285}$$

    where $d^{(2)}\Omega$ refers to the element of solid angle for particle one.

    In the section on angular distributions, we obtained the result for the transformation between these bases in the form:

$$|\theta, \phi, \lambda_1, \lambda_2\rangle = \sum_{j,m} c_j |j, m, \lambda_1, \lambda_2\rangle D^j_{m\alpha}(\phi, \theta, -\phi), \tag{286}$$

    where $\alpha \equiv \lambda_1 - \lambda_2$. Determine the numbers $c_j$.

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