Physics 125
Course Notes
Scattering
Solutions to Problems
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1 Exercises

1. Show that the total cross section we computed in the partial wave expansion,
$$\sigma_T(p) = \frac{4\pi}{p^2}\sum_{j=0}^{\infty}(2j+1)\sin^2\delta_j(p), \tag{1}$$
is in agreement with the optical theorem.

   Solution: Starting with the optical theorem,
$$\begin{aligned}
\sigma_T(p) &= \frac{4\pi}{p}\Im f(p;1) &(2)\\
&= \frac{4\pi}{p}\Im\frac{1}{2ip}\sum_{j=0}^{\infty}(2j+1)\left[e^{2i\delta_j(p)}-1\right]P_j(1) &(3)\\
&= \frac{4\pi}{p^2}\sum_{j=0}^{\infty}(2j+1)\left(-\frac{1}{2}\right)\Re\left[e^{2i\delta_j(p)}-1\right] &(4)\\
&= \frac{4\pi}{p^2}\sum_{j=0}^{\infty}(2j+1)\left[\cos 2\delta_j(p)-1\right] &(5)\\
&= \frac{4\pi}{p^2}\sum_{j=0}^{\infty}(2j+1)\sin^2\delta_j(p). &(6)
\end{aligned}$$

2. We have discussed the "central force problem". Consider a particle of mass m under the influence of the following potential:
$$V(r) = \begin{cases} V_0, & 0 \le r \le a \\ 0, & a < r, \end{cases} \tag{7}$$
where V_0 is a constant.

   (a) Write down the Schrödinger equation for the wave function $\psi(\mathbf{x})$. Consider solutions which are simultaneous eigenvectors of H, $\mathbf{L}^2$, and L_z. Solve the angular dependence, and reduce the remaining problem to a problem in one variable. [You've done this already first quarter, so you may simply retrieve that result here.]

Solution: The Schrödinger equation is

$$\left[-\frac{1}{2m}\nabla^2 + V(r)\right]\psi(\mathbf{x}) = E\psi(\mathbf{x}). \tag{8}$$

The wave function for a state of definite $\mathbf{L}^2 = \ell(\ell+1)$, and $L_z = M$ is $R_{\varepsilon\ell}(r)Y_{\ell M}(\theta, \phi$. The radial wave equation is:

$$\chi_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - k_0^2\right]\chi_\ell = 0, \quad r < a, \tag{9}$$

$$\chi_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]\chi_\ell = 0, \quad r > a, \tag{10}$$

where $\chi_\ell = rR_\ell$ (suppressing the radial index ε), $k^2 = 2mE$, and $k_0^2 = 2mV_0$.

(b) Let E be the eigenvalue of the Hamiltonian, H. Consider the case where $E > V_0$. Solve the Schrödinger equation for eigenstates $\psi(\mathbf{x})$. It will probably be convenient to use the quantity $\kappa = \sqrt{2m(E - V_0)}$. Consider the limit as $r \to \infty$ for your solutions, and give an interpretation in terms of spherical waves.

Solution: Let's use $K = \sqrt{2m(V_0 - E)}$, and do parts (b) and (c) together (hence $K = i\kappa$ in part (b)). For $r < a$ we need a solution for the wave function which is finite at $r = 0$, and for $r > a$ we need something finite at $r = \infty$:

$$\psi(\mathbf{x}) = \sum_{\ell=0}^{\infty} i^\ell(2\ell+1)P_\ell(\cos\theta)\begin{cases} A_\ell j_\ell(iKr), & r < a, \\ j_\ell(kr) + \frac{\alpha}{2}h^{(1)}(kr), & r > a. \end{cases} \tag{11}$$

The constants A_ℓ and α_ℓ may be determined by satisfying the continuity conditions at $r = a$:

$$\lim_{r\to a_-}\psi(\mathbf{x}) = \lim_{r\to a_+}\psi(\mathbf{x}), \tag{12}$$

$$\lim_{r\to a_-}\frac{\partial\psi(\mathbf{x})}{\partial r} = \lim_{r\to a_+}\frac{\partial\psi(\mathbf{x})}{\partial r}. \tag{13}$$

The result is:

$$A_\ell = \frac{j_\ell(ka) + \frac{\alpha}{2}h_\ell^{(1)}(ka)}{j_\ell(iKa)}, \tag{14}$$

and

$$\alpha_\ell = -2\frac{L_\ell j_\ell(ka) - kaj'_\ell(ka)}{L_\ell h_\ell^{(1)}(ka) - kah_\ell^{(1)\prime}(ka)}, \tag{15}$$

where

$$L_\ell = iKa\frac{j'_\ell(iKa)}{j_\ell(iKa)}. \tag{16}$$

Asymptotically,

$$j_\ell(x) \quad \sim_{x\to\infty} \quad \frac{\sin(x-\ell\pi/2)}{x}, \tag{17}$$

$$h_\ell^{(1)}(x) \quad \sim_{x\to\infty} \quad \frac{1}{i^{\ell+1}}\frac{e^{ix}}{x}. \tag{18}$$

See the discussion in section 9 for further interpretation.

(c) Repeat the solution for the case where $E < V_0$. It will probably be convenient to use the quantity $K = \sqrt{2m(V_0 - E)}$. Again, consider the limit as $r \to \infty$ and give an interpretation, contrasting with the previous case.

Solution: See part (b).

Hint: You will probably benefit by thinking about solutions in the form of spherical Bessel/Neumann functions, and/or spherical Hankel functions.

3. When we calculated the density of states for a free particle, we used a "box" of length L (in one dimension), and imposed periodic boundary conditions to ensure no net flux of particles into or out of the box. We have in mind, of course, that we can eventually let $L \to \infty$, and are really interested in quantities per unit length (or volume). Let us justify more carefully the use of periodic boundary conditions, *i.e.*, we wish to convince ourselves that the intuitive rationale given above is correct. To do this, consider a free particle in a one-dimensional "box" from $-L/2$ to $L/2$. Remembering that the Hilbert space of allowed states is a linear space, show that the periodic boundary condition:

$$\psi(-L/2) \quad = \quad \psi(L/2), \tag{19}$$

$$\psi'(-L/2) \quad = \quad \psi'(L/2) \tag{20}$$

is required for acceptable wave functions. "Acceptable" here means that the probability to find a particle in the box must be constant.

Solution: The Schrödinger equation for a free particle is

$$-i\partial_t\psi(x,t) = -\frac{1}{2m}\partial_x^2\psi(x,t). \tag{21}$$

We suppose that an "acceptable" wave function is one which has a constant probability to be in the "box" $(-L/2, L/2)$:

$$\frac{d}{dt}\int_{-L/2}^{L/2}|\psi(x,t)|^2dx = 0. \tag{22}$$

It is readily verified that the function

$$\phi(x,t) = e^{i\frac{2\pi^2}{mL^2}t}\sin\frac{2\pi}{L}x \tag{23}$$

has the desired property.

If we admit $\phi(x,t)$ as an acceptable solution, and if $\psi(x,t)$ is any other acceptable solution, then $\phi+\psi$ must be acceptable, since any linear combination of acceptable solutions must be acceptable. Hence, we must have:

$$\begin{aligned}
\frac{d}{dt}\int_{-L/2}^{L/2}|\psi(x,t)|^2dx &= 0; &(24)\\
\frac{d}{dt}\int_{-L/2}^{L/2}|\phi(x,t)|^2dx &= 0; &(25)\\
\frac{d}{dt}\int_{-L/2}^{L/2}|\psi(x,t)+\phi(x,t)|^2dx &= 0. &(26)
\end{aligned}$$

Then we may write (assuming Eqns 24 and 25):

$$\begin{aligned}
0 &= \frac{d}{dt}\int_{-L/2}^{L/2}\left[\psi(x,t)\phi^*(x,t)+\psi^*(x,t)\phi(x,t)\right]dx &(27)\\
&= \int_{-L/2}^{L/2}\partial_t\left[\psi(x,t)\phi^*(x,t)+\psi^*(x,t)\phi(x,t)\right]dx &(28)\\
&= \frac{i}{2m}\int_{-L/2}^{L/2}\left[\left(\partial_x^2\psi\right)\phi^* - \psi\left(\partial_x^2\phi\right) + \psi^*\left(\partial_x^2\phi\right) - \left(\partial_x^2\psi^*\right)\phi\right]dx &(29)\\
&= \int_{-L/2}^{L/2}\partial_x\left[\left(\partial_x\psi\right)\phi^* - \psi\left(\partial_x\phi\right) + \psi^*\left(\partial_x\phi\right) - \left(\partial_x\psi^*\right)\phi\right]dx &(30)\\
&= \left[\left(\partial_x\psi\right)\phi^* - \psi\left(\partial_x\phi\right) + \psi^*\left(\partial_x\phi\right) - \left(\partial_x\psi^*\right)\phi\right]_{-L/2}^{L/2}. &(31)
\end{aligned}$$

But $\phi(\pm L/2, t) = 0$, so

$$0 = [-\psi(\partial_x \phi^*) + \psi^*(\partial_x \phi)]_{-L/2}^{L/2}. \tag{32}$$

Further, since

$$\partial_x \phi(\pm L/2, t) = -\frac{2\pi}{L} e^{i\frac{2\pi^2}{mL^2}t}, \tag{33}$$

we obtain

$$0 = \psi(L/2,t)e^{-i\frac{2\pi^2}{mL^2}t} - \psi(-L/2,t)e^{-i\frac{2\pi^2}{mL^2}t} + \psi^*(L/2,t)e^{i\frac{2\pi^2}{mL^2}t} - \psi^*(-L/2,t)e^{i\frac{2\pi^2}{mL^2}t}. \tag{34}$$

This must be true for all times; also if ψ is acceptable, then $e^{i\theta}\psi$ must be acceptable, for real θ. Hence, ψ is acceptable if and only if Eqn. 24 holds, and:

$$\psi(L/2, t) = \psi(-L/2, t). \tag{35}$$

We note that the function $e^{i\frac{2\pi^2}{mL^2}t} \cos \frac{2\pi}{L}x$ satisfies these criteria. Thus, we could also have picked

$$\phi(x, t) = e^{i\frac{2\pi^2}{mL^2}t} \cos \frac{2\pi}{L}x \tag{36}$$

as an acceptable solution. Then the same argument reveals that any other acceptable solution ψ must satisfy the boundary condition:

$$\partial_x \psi(L/2, t) = \partial_x \psi(-L/2, t). \tag{37}$$

We finally remark that the set of functions $\left\{e^{i\frac{n^2 2\pi^2}{mL^2}t} \sin \frac{2\pi n}{L}x, e^{i\frac{n^2 2\pi^2}{mL^2}t} \cos \frac{2\pi n}{L}x; n = 0, 1, \ldots\right\}$ is a complete set of functions with the required boundary conditions.

4. In our discussion of scattering theory, we supposed we had a beam of particles from some ensemble of wave packets, and obtained an "effective" (observed) differential cross-section:

$$\sigma_{\text{eff}}(\mathbf{u}) = \int_{\{\alpha\}} f(\alpha)d\alpha \int_{|\mathbf{x}|\le R} d^2(\mathbf{x}) P(\mu; \propto; \mathbf{x})$$

This formula assumed that the beam particles were distributed uniformly in a disk of radius R centered at the origin in the $\hat{e}_1 - \hat{e}_2$ plane, and that the distribution of the shape parameter was uncorrelated with position in this disk.

(a) Try to obtain an expression for $\sigma_{\text{eff}}(\mathbf{u})$ without making these assumptions.

Solution: We start with Eqn. 33 from the note:

$$P(\mathbf{u};\alpha;\mathbf{x}) = \int_{(\infty)} d^3(\mathbf{q}) \int_{(\infty)} d^3(\mathbf{q}') q^2 \delta(q-q') \quad (38)$$

$$T(q\mathbf{u},\mathbf{q})T^*(q'\mathbf{u},\mathbf{q}')\phi_0(\mathbf{q};\alpha)\phi_0^*(\mathbf{q}';\alpha)e^{-i\mathbf{x}\cdot(\mathbf{q}-\mathbf{q}')} \quad (39)$$

In general, if $f(\alpha,\mathbf{x})$ describes the beam position and shape distribution (possibly correlated), with

$$\int_{\{\alpha\}} \int_{(\infty))} f(\alpha,\mathbf{x}) d\alpha d^2(\mathbf{x}) = 1, \quad (40)$$

then the effective differential cross section is:

$$\frac{d\sigma_{\text{eff}}(\mathbf{u})}{d\omega} = A_{\text{eff}} \int_{\{\alpha\}} \int_{(\infty))} f(\alpha,\mathbf{x}) P(\mathbf{u};\alpha;\mathbf{x}) d\alpha d^2(\mathbf{x}), \quad (41)$$

where A_{eff} is an "effective" area of the beam.

The effective area of the beam may be computed by requiring that we get a consistent answer for a small "hard" target, of area a. In this case, $P = 1$ for $|\mathbf{x}| < \sqrt{a/\pi}$. Thus,

$$a = A_{\text{eff}} \int_{\{\alpha\}} \int_{|\mathbf{x}|<\sqrt{a/\pi}} f(\alpha,\mathbf{x}) d\alpha d^2(\mathbf{x}). \quad (42)$$

We want this equality to hold in the limit as $a \to 0$:

$$A_{\text{eff}} = \lim_{a\to 0} \frac{a}{\int_{\{\alpha\}} d\alpha \int_0^{\sqrt{a/\pi}} r dr \int_0^{2\pi} f(\alpha,\mathbf{x})}. \quad (43)$$

(b) Using part (a), write down an expression for $\sigma_{\text{eff}}(\mathbf{u})$ appropriate to the case where the beam particles are distributed according to a Gaussian of standard deviation ρ in radial distance from the origin (in the $\hat{e}_2 - \hat{e}_3$ plane), and where the wave packets are also drawn from a Gaussian distribution in the expectation value of the magnitude of the momentum. Let the standard deviation of this momentum distribution be $\alpha = \alpha(\mathbf{x})$, for beam position $\mathbf{x}$.

Solution: We have a beam distribution:

$$f(p,\mathbf{x}) = \frac{1}{2\pi\rho^2}e^{-r^2/2\rho^2}\frac{1}{\sqrt{2\pi}\alpha(\mathbf{x})}e^{-(p-p_0)^2/2\alpha^2(\mathbf{x})}. \tag{44}$$

The effective area is:

$$\begin{aligned}
A_{\text{eff}} &= \lim_{a\to 0}\frac{a}{\int_0^{\sqrt{a/\pi}} rdr\int_0^{2\pi} d\phi\int_{-\infty}^{\infty} dp\frac{1}{2\pi\rho^2}e^{-r^2/2\rho^2}\frac{1}{\sqrt{2\pi}\alpha(\mathbf{x})}e^{-(p-p_0)^2/2\alpha^2(\mathbf{x})}} \\
&= \lim_{a\to 0}\frac{a}{\int_0^{\sqrt{a/\pi}} rdr2\pi\frac{1}{2\pi\rho^2}e^{-r^2/2\rho^2}} \\
&= \lim_{a\to 0}\frac{2a}{\int_0^{\sqrt{a/\pi}} dr^2\frac{1}{\rho^2}e^{-r^2/2\rho^2}} \\
&= \lim_{a\to 0}\frac{2a}{\int_0^{a/\pi} dr^2\frac{1}{\rho^2}e^{-r^2/2\rho^2}} \\
&= \lim_{a\to 0}\frac{a}{1-e^{-a^2/2\pi\rho^2}} \\
&= 2\pi\rho^2. \qquad (45)
\end{aligned}$$

Thus,

$$\frac{d\sigma_{\text{eff}}(\mathbf{u})}{d\omega} = 2\pi\rho^2\int_{(\infty))} d^2(\mathbf{x})\int dp f(p,\mathbf{x})P(\mathbf{u};p;\mathbf{x})d^2(\mathbf{x}). \tag{46}$$

(c) For your generalized result of part (a), try to repeat our limiting case argument to obtain the "fundamental" cross section. Discuss.

Solution: The limiting case corresponds to the beam being spread out over a size large compared with the target, and with a sharply defined momentum. The same arguments as in the note will hence apply.

5. Let us briefly consider the consequences of reflection invariance (parity conservation) for the scattering of a particle with spin s on a spinless target. [We consider elastic scattering only here]. Thus, assume the interaction is reflection invariant:

   (a) How does the S matrix transform under parity, *i.e.*, what is $P^{-1}SP$, where P is the parity operator?

Solution: If the interaction is invariant under reflection, then

$$P^{-1}SP = S. \tag{47}$$

(b) What is the condition on the helicity amplitudes $A^j_{\lambda\mu}(p_i)$ (corresponding to scattering with total angular momentum j) imposed by parity conservation?

Solution: Under parity, the helicity λ reverses sign to $-\lambda$. Hence,

$$P|p; j\lambda\rangle = (-)^{\eta}_{\text{intrinsic}}|p; j\lambda\rangle. \tag{48}$$

The helicity amplitude is:

$$A^j_{\lambda\mu}(p_i) = \langle p_i; jm, \lambda|S|p_i; jm, \mu\rangle \tag{49}$$

Under parity,

$$A^j_{\lambda\mu}(p_i) \to A^j_{-\lambda-\mu}(p_i) \tag{50}$$

Thus, parity conservation requires

$$A^j_{\lambda\mu}(p_i) = A^j_{-\lambda-\mu}(p_i) \tag{51}$$

(c) What condition is imposed on the orbital angular momentum amplitudes $B^j_{\ell\ell'}(p_i)$? You may use "physical intuition" if you like, but it should be convincing. In any event, be sure your answer makes intuitive sense.

Solution: Under a parity transformation, a wave function corresponding to definite orbital angular momentum ℓ transforms as

$$P\psi_\ell(\mathbf{x}) = (-)^\ell\psi_\ell(\mathbf{x}). \tag{52}$$

The orbital angular momentum amplitude is:

$$\begin{aligned} B^j_{\ell\ell'}(p_i) &= \langle p_i; jm, \ell|S|p_i; jm, \ell'\rangle & (53)\\ &= \langle p_i; jm, \ell|P^\dagger PSP^\dagger P|p_i; jm, \ell'\rangle & (54)\\ &= (-)^{\ell-\ell'}\langle p_i; jm, \ell|S|p_i; jm, \ell'\rangle. & (55) \end{aligned}$$

If parity is conserved, then $B^j_{\ell\ell'}(p_i) = 0$ if $\ell - \ell'$ is odd.

6. We consider the resonant scattering of light by an atom. In particular, let us consider sodium, with ${}^2P_{1/2} \leftrightarrow {}^2 S_{1/2}$ resonance at $\lambda = \lambda_0 =$ 5986Å. Let σ_{0T} be the total cross section at resonance, for a monochromatic light source (*i.e.*, σ_{0T} is the "fundamental" cross section).

   (a) Ignoring spin, estimate σ_{0T}, first in terms of $\lambda_0/2\pi$, and then numerically in cm^2. Compare your answer with a typical atomic size.

   Solution: The wavelength and photon momentum are related by $\lambda = 2\pi/p$, or $\lambdabar = \lambda/2\pi = 1/p$. The total cross section on a resonance in partial wave ℓ is:

$$\sigma_{\ell T}(E) = \frac{4\pi}{p^2}(2\ell+1)\frac{\Gamma^2/4}{(E-E_0)^2+\Gamma^2/4}. \tag{56}$$

   The wavelength here is much larger than the atom, so we presume this to be an S-wave resonance (as suggested by the "0" in the problem statement). Hence, the cross section at the resonance peak is:

$$\begin{aligned}\sigma_{0T}(E_0) &= \frac{4\pi}{p^2} = \lambda^2/\pi = 4\pi\lambdabar^2 &(57)\\ &= 1.14\times 10^{-9}\ \text{cm}^2. &(58)\end{aligned}$$

   A typical atomic size is of order $\mathring{A}^2$, or $\sim 10^{-15}$ cm^2, which is much smaller than this resonant cross section.

   (b) Suppose that we have a sodium lamp source with a line width governed by the mean life of the excited ${}^2P_{1/2}$ state (maybe not easy to get this piece of equipment!). The mean life of this state is about 10^{-8} second. Suppose that this light is incident on an absorption cell, containing sodium vapor and an inert (non-resonant) buffer gas. Let the temperature of the gas in the absorption cell be 200°C. Obtain an expression for the effective total cross section, $\sigma_{\text{eff}T}$ which an atom in the cell presents to the incident light. Again, make a numerical calculation in cm^2.

   Solution: The line width of our light source is

$$\Gamma \approx 1/10^{-8} = 10^8\ \text{Hz}. \tag{59}$$

The sodium atoms will move thermally according to a Maxwell-Boltzmann distribution:

$$p(E) = Ae^{-E/k_BT} \tag{60}$$

The Doppler broadening of the absorption line in the sodium vapor cell, due to thermal motion, is

$$\begin{aligned} \Delta\nu &= 2\nu_0(2k_BT\ln 2/m)^{1/2} &(61)\\ &= 1.5\times 10^9 \text{ Hz.} &(62) \end{aligned}$$

The atoms thus see a gaussianly distributed line-width:

$$f(\nu) = \frac{2(\ln 2)^{1/2}}{\sqrt{\pi}\Delta\nu_0} e^{-4\ln 2(\nu-\nu_0)^2/\Delta\nu_0^2}. \tag{63}$$

In principle, we take a convolution of this with the resonant line shape. However, the gaussian is relatively wide, so we may approximate it with its value at ν_0:

$$\sigma_{\text{eff}_T} \approx \frac{4\pi}{k^2} f(\nu_0)\Gamma \approx 6.6\times 10^{-11} \text{ cm}^2. \tag{64}$$

(c) Using your result above, find the number density of Na atoms (# of atoms/cm^3) which is required in the cell in order that intensity of the incident light is reduced by a factor of two in a distance of 1 cm. It should be noted (and your answer should be plausible here) that such a gas will be essentially completely transparent to light of other (non-resonant) wavelengths.

Solution: The attenuation of the light is exponential:

$$I(L) = I(0)e^{-N\sigma_{\text{eff}}L}, \tag{65}$$

where N is the number density. We want the density such that $I(1\,\text{cm})/I(0) = 1/2$:

$$N = \frac{\ln 2}{\sigma_{\text{eff}}L}. \tag{66}$$

7. Consider scattering from the simple potential:

$$V(\mathbf{x}) = \begin{cases} V_0 & r = |\mathbf{x}| < R \\ 0 & r > R. \end{cases}$$

In the low energy limit, we might only look at S-wave $\ell = 0$ scattering. However, in the high energy limit, we expect scattering in other partial waves to become significant. For simplicity, let us here consider scattering on a hard sphere, $V_0 \to \infty$.

(a) For a hard sphere potential, calculate the total cross section in partial wave ℓ. Give the exact result, *i.e.*, don't take the high energy limit yet. You may quote your answer in terms of the spherical Bessel functions.

Solution: The total cross section is

$$\sigma_T = \sum_{\ell=0}^{\infty} \sigma_l, \tag{67}$$

where σ_ℓ is the total cross section in partial wave ℓ.

The problem has azimuthal symmetry, taking the incident wave to be along the z axis, so the angular solutions may be expressed in terms of the Legendre polynomials. The radial wave equation, for $r > R$, may be expressed as

$$\chi'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]\chi_\ell = 0. \tag{68}$$

The solution to the Schrödinger equation for $r > R$ is:

$$\psi(\mathbf{x}) = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\left[j_\ell(kr) + \frac{1}{2}\alpha_\ell h_\ell^{(1)}(kr)\right] P_\ell(\cos\theta). \tag{69}$$

If we have a hard sphere potential, then the boundary condition is that $\psi(r = R, \Omega) = 0$. Hence,

$$j_\ell(kR) + \frac{1}{2}\alpha_\ell h_\ell^{(1)}(kR) = 0, \quad \ell = 0, 1, \dots \tag{70}$$

Therefore, for the hard sphere potential,

$$\alpha_\ell(k) = e^{2i\delta_\ell} - 1 = -2\frac{j_\ell(kR)}{h_\ell^{(1)}(kR)}. \tag{71}$$

For $\ell = 0$, this reduces to $\alpha_0 = e^{-2ikR} - 1$, or $\delta_0 = -kR$. This is also the result for $\ell = 1$.

The total cross section in partial wave ℓ is

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell \tag{72}$$
$$= \frac{4\pi}{k^2}(2\ell+1)\left|\frac{e^{2i\delta_\ell}-1}{2i}\right|^2 \tag{73}$$
$$= \frac{4\pi}{k^2}(2\ell+1)|\alpha_\ell/2|^2 \tag{74}$$
$$= \frac{4\pi}{k^2}(2\ell+1)\left|\frac{j_\ell(kR)}{h_\ell^{(1)}(kR)}\right|^2 \tag{75}$$
$$= \frac{4\pi}{k^2}(2\ell+1)\frac{[j_\ell(kR)]^2}{[j_\ell(kR)]^2+[n_\ell(kR)]^2}, \tag{76}$$

where, in the final step we have used $h_\ell^{(1)}(x) = j_\ell(x) + in_\ell(x)$.

(b) Find a simple expression for the phase shift δ_ℓ in the high energy limit ($kR \gg \ell$). Keep terms up to O(1) in your result.

Solution: At high energies, letting $x = kR$:

$$\alpha_\ell(k) = e^{2i\delta_\ell} - 1 = -2\frac{j_\ell(kR)}{h_\ell^{(1)}(kR)} \tag{77}$$
$$\approx -2\cos\left(x - \ell\frac{\pi}{2} - \frac{\pi}{2}\right)e^{-i\left(x-\ell\frac{\pi}{2}-\frac{\pi}{2}\right)} \tag{78}$$
$$= e^{-2i\left(x-\ell\frac{\pi}{2}-\frac{\pi}{2}\right)} - 1. \tag{79}$$

Thus, for $kR \gg \ell$,

$$\delta_\ell = -kR + \ell\frac{\pi}{2} + \frac{\pi}{2}. \tag{80}$$

(c) Determine the total cross section (including all partial waves) in the high energy limit, $kR \to \infty$. [This is the only somewhat tricky part of this problem to calculate. One approach is as follows: Write down the total cross section in terms of your results for part (a). Then, for fixed k, consider which values of ℓ may be important in the sum. Neglect the other values of ℓ, and make the high energy approximation to your part (a) result. Finally, evaluate the sum, either directly, or by turning it into an appropriate integral.]

Solution: We must evaluate:

$$\sigma_T = 4\pi R^2 \lim_{x=kR\to\infty} \frac{1}{x^2}\sum_{\ell=0}^{\infty}(2\ell+1)\frac{[j_\ell(x)]^2}{[j_\ell(x)]^2+[n_\ell(x)]^2}. \tag{81}$$

We make use of the following facts:

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}J_{\ell+\frac{1}{2}}(x) \tag{82}$$

$$n_\ell(x) = \sqrt{\frac{\pi}{2x}}Y_{\ell+\frac{1}{2}}(x) \tag{83}$$

$$h_\ell^{(1)}(x) = \sqrt{\frac{\pi}{2x}}H^{(1)}_{\ell+\frac{1}{2}}(x) \tag{84}$$

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\cos(x-\nu\frac{\pi}{2}-\frac{\pi}{4}) \tag{85}$$

$$H^{(1)}_\nu(x) \sim \sqrt{\frac{2}{\pi x}}e^{i(x-\nu\frac{\pi}{2}-\frac{\pi}{4})}. \tag{86}$$

Thus, with $\nu=\ell+1/2$,

$$\left|\frac{j_\ell(x)}{h_\ell^{(1)}(x)}\right|^2 \sim \left|\frac{\cos(x-\ell\frac{\pi}{2}-\frac{\pi}{2})}{e^{i(x-\ell\frac{\pi}{2}-\frac{\pi}{2})}}\right|^2 \tag{87}$$

$$\sim \cos^2(x-\ell\frac{\pi}{2}-\frac{\pi}{2}) \tag{88}$$

$$\sim \sin^2(x-\ell\frac{\pi}{2}), \quad \text{for } x\gg\ell. \tag{89}$$

We further note that, for fixed x, $j_\ell(x)$ approaches zero for large ℓ, and $h_\ell(x)$ approaches infinity for large ℓ. Let us argue that we may cut off the sum at $\ell = kR$ on physical grounds: At high energy, $1/k \ll R$. Now $\ell \sim kr$, since ℓ is the orbital angular momentum quantum number. If $\ell > kR$, then $r > R$, and the short wavelength beam misses the target, hence there is no contribution to the scattering cross section. Thus, in the high energy limit, for scattering on a hard sphere:

$$\sigma_T = 4\pi R^2 \lim_{x=kR\to\infty}\frac{1}{x^2}\sum_{\ell=0}^{x}(2\ell+1)\sin^2(x-\ell\frac{\pi}{2}). \tag{90}$$

Now

$$\sin^2(x-\ell\frac{\pi}{2}) = \begin{cases}\sin^2 x, & \ell \text{ even},\\ \cos^2 x, & \ell \text{ odd}.\end{cases} \tag{91}$$

We evaluate the sum in this limit:

$$\begin{aligned}\frac{1}{x^2}\sum_{\ell=0}^{x}(2\ell+1)\sin^2(x-\ell\frac{\pi}{2}) &= \sin^2 x+\cos^2 x+2\sin^2 x+2\cos^2 x \\ &\quad +3\sin^2 x+3\cos^2 x+\ldots \quad (92)\\ &= \sum_{\ell=0}^{x}\ell = \frac{1}{2}x(x+1). \quad (93)\end{aligned}$$

Hence,

$$\sigma_T \sim \frac{4\pi}{k^2}\sum_{\ell=0}^{kR}\ell = \frac{4\pi}{k^2}\left(\frac{k^2R^2}{2}\right) = 2\pi R^2. \quad (94)$$

8. Consider the graph in Fig. 1.

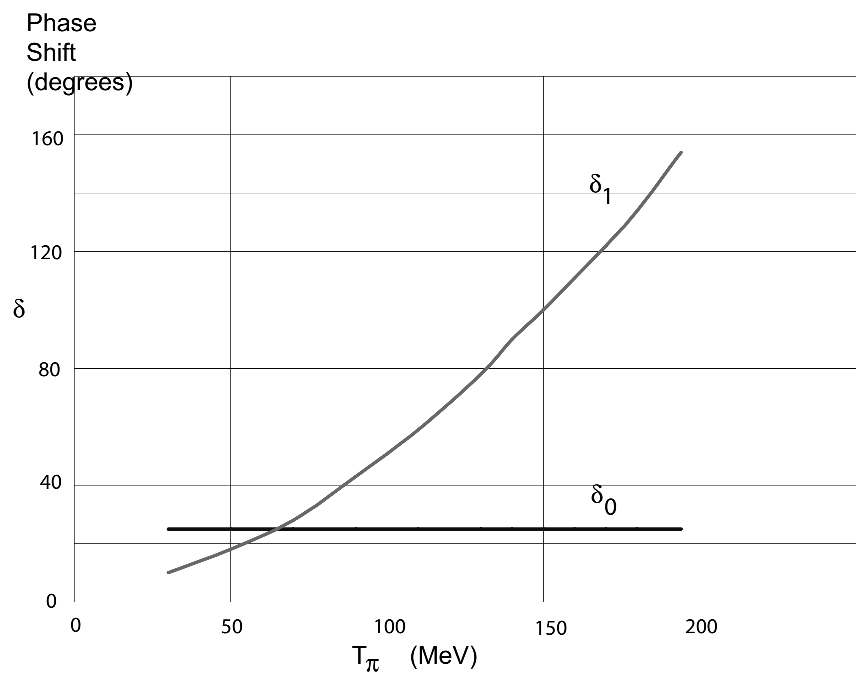

Figure 1: Made-up graph of phase shifts δ_0 and δ_1 for elastic π^+p scattering (neglecting spin).

Assume that the other phase shifts are negligible (*e.g.*, "low energy" is reasonably accurate). The pion mass and energy here are sufficiently small that we can at least entertain the approximation of an infinitely heavy proton at rest – we'll assume this to be the case, in any event. Note that T_π is the relativistic kinetic energy of the $\pi^+ : T_\pi = \sqrt{P_\pi^2 + m_\pi^2} - m_\pi$.

(a) Is the $\pi^+ p$ force principally attractive or repulsive (as shown in this figure)?

Solution: The phase shifts are positive, indicating a dominantly attractive potential.

(b) Plot the total cross section in mb (millibarns) as a function of energy, from T_π=40 to 200 MeV.

Solution: The total cross section in terms of the partial wave phase shifts is:

$$\sigma_T = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell \tag{95}$$

$$= \frac{4\pi}{k^2}(\sin^2\delta_0 + 3\sin^2\delta_1). \tag{96}$$

The kinetic energy T_π is related to k by $T_\pi = \sqrt{m_\pi^2 + k^2} - m_\pi$, or

$$k = \sqrt{T(T+2m_\pi)}. \tag{97}$$

To convert to millibarns, we multiply by:

$$1 = (197\,\text{MeV-fm})^2 10\,\text{mb/fm}^2 = 3.88\times 10^5\,\text{MeV}^2\text{mb}. \tag{98}$$

Figure 2 shows the result.

(c) Plot the angular distribution of the scattered π^+ at energies of 120, 140 and 160 MeV.

Solution:

$$\frac{d\sigma}{d\Omega} = |\frac{1}{2ik}\sum_{j=0}^{\infty}(2j+1)\left[e^{2i\delta_j(k)}-1\right]P_j(\cos\theta)|^2 \tag{99}$$

$$= \frac{1}{4k^2}|e^{2i\delta_0(k)} - 1 + 3(e^{2i\delta_j(k)}-1)\cos\theta|^2 \tag{100}$$

$$= \frac{1}{4k^2}\left\{[\cos\delta_0 - 1 + 3(\cos\delta_1 - 1)\cos\theta]^2 + [\sin\delta_0 + 3\sin\delta_1\cos\theta]^2\right\}.$$

The result is shown in Fig. 3.

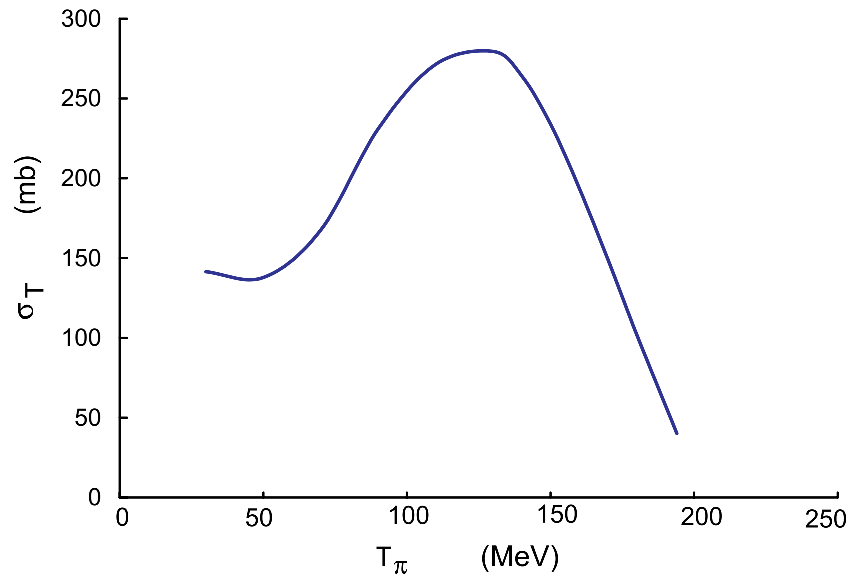

Figure 2: Total (made-up) cross section for elastic π^+p scattering (neglecting spin).

(d) What is the mean free path of 140 MeV pions in a liquid hydrogen target, with these "protons"?

Solution: The cross section for 140 MeV pions is ~ 260 mb. The density of liquid hydrogen is 0.0708 g/cm^3. The number density is $\rho == 4.2 \times 10^28\,\mathrm{m}^{-3}$. The mean free path is thus

$$\lambda = \frac{1}{\sigma_T \rho} = 0.9\,\mathrm{m}. \tag{101}$$

9. We now start to consider the possibility of "inelastic scatting". For example, let us suppose there is a "multiplet" of N non-identical particles, all of mass m. We consider scattering on a spherically symmetric center-of-force, with the properly that the interaction can change a particle from one number of the multiplet to another member. We may in this case express the scattering amplitude by $f_{\alpha\beta}(k;\cos\theta)$, with $\alpha, \beta = 1, \ldots, N$, corresponding to a unitary S-matrix on Hilbert space $L_2(R^3) \otimes V_N$:

$$S_{\alpha\beta}(\mathbf{k}_f;\mathbf{k}_i) = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{k}_f - \mathbf{k}_i) + \frac{i}{2\pi k_i}\delta(k_f - k_i) f_{\alpha\beta}(\mathbf{k}_f;\mathbf{k}_i)$$

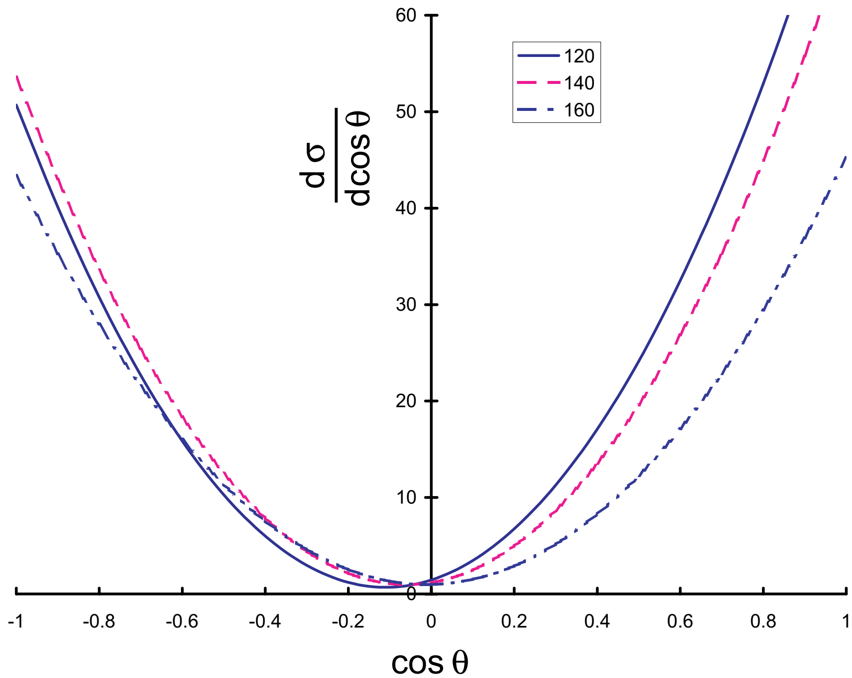

Figure 3: Differential (made-up) cross section for elastic π^+p scattering (neglecting spin), at three values of T_π.

β is here to be interpreted as identifying the initial particle, and α the final particle. The generalization of our partial wave expansion to this situation is clearly:

$$f_{\alpha\beta}(k;\cos\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(A^{(\ell)}_{\alpha\beta} - \delta_{\alpha\beta})P_\ell(\cos\theta)$$

Where $A^{(\ell)}$ is an $N \times N$ unitary matrix:

$$A^{(\ell)} = \exp[2i\triangle_\ell(k)]$$

with $\triangle_\ell$ an $N \times N$ Hermitian "phase shift matrix" note that $f_{\alpha\alpha}$ is the elastic scattering amplitude for particle α.

(a) Find expressions, in terms of $A^{(\ell)}(k)$, for the following total cross sections, for an incident particle α: (integrated over angles)

i. $\sigma^{\rm el}_{\alpha\rm TOT}$, the total elastic cross section
Solution: We'll use

$$\int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}. \tag{102}$$

$$\begin{aligned}
\sigma^{\rm el}_{\alpha\rm TOT} &= 2\pi \int_{-1}^{1} d\cos\theta |f_{\alpha\alpha}(k;\cos\theta)|^2 &(103)\\
&= \frac{\pi}{2k^2}\sum_\ell (2\ell+1)^2 \int_{-1}^{1} dx |P_\ell(x)|^2 |A^{(\ell)}_{\alpha\alpha} - 1|^2 &(104)\\
&= \frac{\pi}{k^2}\sum_\ell (2\ell+1)|A^{(\ell)}_{\alpha\alpha} - 1|^2. &(105)
\end{aligned}$$

ii. $\sigma^{\rm inel}_{\alpha\rm TOT}$, the total inelastic cross section (sometimes called the "reaction" cross section).
Solution:

$$\begin{aligned}
\sigma^{\rm inel}_{\alpha\rm TOT} &= 2\pi \sum_{\beta\neq\alpha}\int_{-1}^{1} d\cos\theta |f_{\beta\alpha}(k;\cos\theta)|^2 &(106)\\
&= \frac{\pi}{k^2}\sum_{\beta\neq\alpha}\sum_\ell (2\ell+1)|A^{(\ell)}_{\beta\alpha}|^2. &(107)
\end{aligned}$$

iii. $\sigma_{\alpha\rm TOT}$, the total cross section.
Solution:

$$\begin{aligned}
\sigma_{\alpha\rm TOT} &= \sigma^{\rm el}_{\alpha\rm TOT} + \sigma^{\rm inel}_{\alpha\rm TOT} &(108)\\
&= \frac{\pi}{k^2}\sum_{\beta}\sum_\ell (2\ell+1)|A^{(\ell)}_{\beta\alpha} - \delta_{\alpha\beta}|^2. &(109)
\end{aligned}$$

(b) Try to give the generalization of the optical theorem for this scattering of particles in a multiplet.
Solution: Start with the unitarity os the S matrix:

$$\sum_\beta \int_{(\infty)} d^3(\mathbf{q}) S_{\alpha\beta}(\mathbf{p}', \mathbf{q}) S^\dagger_{\gamma\beta}(\mathbf{q}, \mathbf{p}'') = \delta_{\alpha\gamma}\delta^{(3)}(\mathbf{p}' - \mathbf{p}''). \tag{110}$$

Substitute in our form for S in terms of the scattering amplitude f, and arrive at:

$$\sum_\gamma \Im f_{\alpha\gamma}(p;1) = \frac{p}{4\pi}\sigma_\alpha\text{TOT}. \tag{111}$$

10. In the previous problem you considered the scattering of particles in a multiplet. You determined the total elastic (sometimes called "scattering") cross section and the total inelastic ("reaction") cross sections in terms of the $A^{(\ell)}_{\alpha\beta}$ matrix in the partial wave expansion. Consider now the graph in Fig. 4.

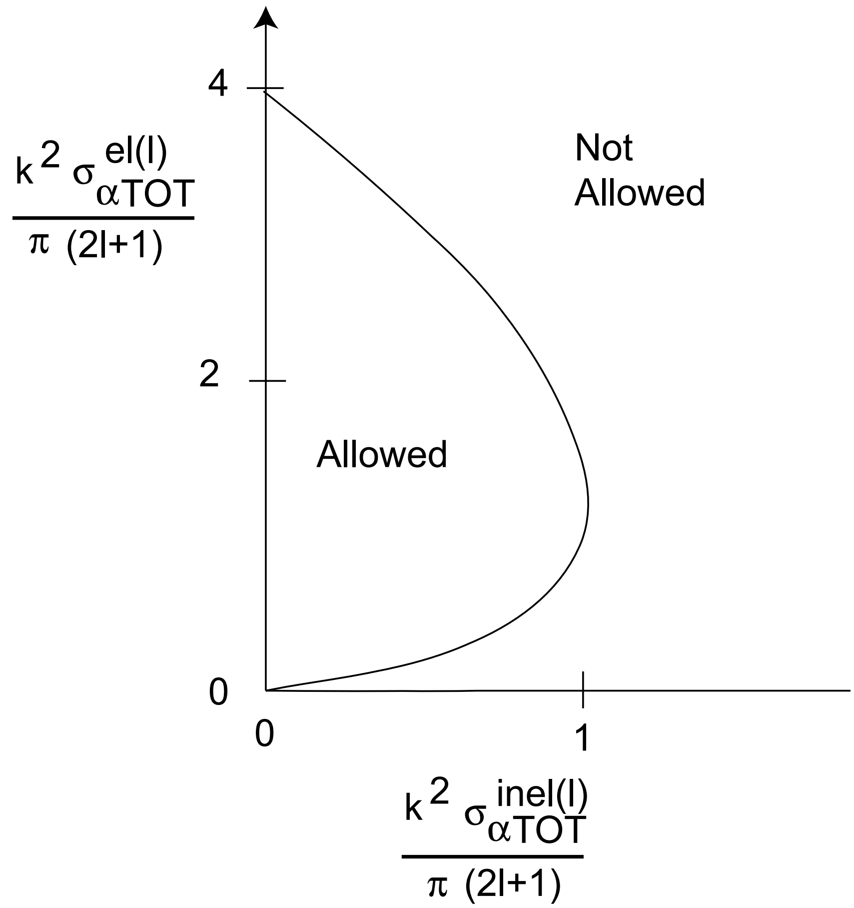

Figure 4: The allowed and forbidden regions for possible elastic and inelastic cross sections for the scattering of particles in a multiplet.

This graph purports to show the allowed and forbidden regions for the total elastic and inelastic cross sections in a given partial wave ℓ. Derive the formula for the allowed region of this graph. Make sure to check the extreme points.

Solution: For simplicity, let the vertical axis be v, and the horizontal axis u:

$$u = \frac{k^2\sigma_{\alpha\mathrm{TOT}}^{\mathrm{inel}(\ell)}}{\pi(2\ell+1)}; v = \frac{k^2\sigma_{\alpha\mathrm{TOT}}^{\mathrm{el}(\ell)}}{\pi(2\ell+1)}. \tag{112}$$

From the solution to the previous problem, and unitarity of the $A^{(\ell)}$ matrix, we thus have

$$u = \sum_{\beta\neq\alpha} |A^{(\ell)}_{\beta\alpha}|^2 = 1 - |A^{(\ell)}_{\alpha\alpha}|^2, \tag{113}$$

$$v = |A^{(\ell)}_{\alpha\alpha} - 1|^2 = 1 + |A^{(\ell)}_{\alpha\alpha}|^2 - 2\Re A^{(\ell)}_{\alpha\alpha}. \tag{114}$$

The constraint imposed by unitarity is that $|A^{(\ell)}_{\alpha\alpha}|^2 \leq 1$. Let $A^{(\ell)}_{\alpha\alpha} = re^{i\theta}$. Then $r \leq 1$ and $0 \leq \theta < 2\pi$ gives the allowed region. In terms of the plotted quantities, $u = 1 - r^2$ and $v = 1 + r^2 - 2r\cos\theta$. Thus

$$0 \leq u \leq 1, \tag{115}$$

and for given u, v must be in the range

$$(1-r)^2 \leq y \leq (1+r)^2, \tag{116}$$

where $r = \sqrt{1-x}$. If $r = 0$ then $(u,v) = (1,1)$. If $r = 1$ then $u = 0$ and $0 \leq v \leq 4$.

11. In the angular distribution section, we discussed the transformation between two different types of "helicity bases". In particular, we considered a system of two particles, with spins j_1 and j_2, in their CM frame.

One basis is the "spherical helicity basis", with vectors of the form:

$$|j, m, \lambda_1, \lambda_2\rangle, \tag{117}$$

where j is the total angular momentum, m is the total angular momentum projection along the 3-axis, and λ_1, λ_2 are the helicities of the two particles. We assumed a normalization of these basis vectors such that:

$$\langle j', m', \lambda_1', \lambda_2' | j, m, \lambda_1, \lambda_2\rangle = \delta_{jj'}\delta_{mm'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}. \tag{118}$$

The other basis is the "plane-wave helicity basis", with vectors of the form:

$$|\theta, \phi, \lambda_1, \lambda_2\rangle, \tag{119}$$

where θ and ϕ are the spherical polar angles of the direction of particle one. We did not specify a normalization for these basis vectors, but an obvious (and conventional) choice is:

$$\langle \theta', \phi', \lambda_1', \lambda_2' | \theta, \phi, \lambda_1, \lambda_2 \rangle = \delta^{(2)}(\Omega' - \Omega)\delta_{\lambda_1 \lambda_1'}\delta_{\lambda_2 \lambda_2'}, \tag{120}$$

where $d^{(2)}\Omega$ refers to the element of solid angle for particle one.

In the section on angular distributions, we obtained the result for the transformation between these bases in the form:

$$|\theta, \phi, \lambda_1, \lambda_2\rangle = \sum_{j,m} c_j |j, m, \lambda_1, \lambda_2\rangle D^j_{m\alpha}(\phi, \theta, -\phi), \tag{121}$$

where $\alpha \equiv \lambda_1 - \lambda_2$. Determine the numbers c_j.

Solution: To select a particular c_j, *i.e.*, a particular j, let us invert the basis transformation:

$$\int_{4\pi} d\Omega D^{j*}_{m\delta}(\phi, \theta, -\phi)|\theta, \phi, \lambda_1, \lambda_2\rangle \quad = \tag{122}$$

$$\sum_{j',m'} c_{j'} |j', m', \lambda_1, \lambda_2\rangle \int_{4\pi} d\Omega D^{j*}_{m\alpha}(\phi, \theta, -\phi) D^{j'}_{m'\alpha}(\phi, \theta, -\phi)$$

$$= \sum_{j',m'} c_{j'} |j', m', \lambda_1, \lambda_2\rangle \int_{4\pi} d\Omega d^j_{m\delta}(\theta) d^{j'}_{m'\alpha}(\theta) \exp\left[-i(m'\phi - \alpha\phi) + i(m\phi - \alpha\phi)\right]$$

$$= \sum_{j',m'} c_{j'} |j', m', \lambda_1, \lambda_2\rangle \int_{-1}^{1} d\cos\theta d^j_{m\alpha}(\theta) d^{j'}_{m'\delta}(\theta) \int_0^{2\pi} d\phi e^{i(m-m')\phi} \tag{123}$$

$$= 2\pi \sum_{j'} c_{j'} |j', m, \lambda_1, \lambda_2\rangle \int_{-1}^{1} d\cos\theta d^j_{m\alpha}(\theta) d^{j'}_{m\alpha}(\theta) \tag{124}$$

$$= 2\pi \sum_{j'} c_{j'} |j', m, \lambda_1, \lambda_2\rangle \frac{2\alpha_{jj'}}{2j+1} \tag{125}$$

$$= \frac{4\pi}{2j+1} c_j |j, m, \lambda_1, \lambda_2\rangle. \tag{126}$$

Note that we should perhaps justify the interchange of the order of summation and integration in the very first step above. Thus,

$$|j, m, \lambda_1, \lambda_2\rangle = \frac{2j+1}{4\pi b_j} \int_{4\pi} d\Omega D^{j*}_{m\alpha}(\phi, \theta, -\phi)|\theta, \phi, \lambda_1, \lambda_2\rangle. \tag{127}$$

Now,

$$
\begin{aligned}
1 &= \langle j, m, \lambda_1, \lambda_2 | j, m, \lambda_1, \lambda_2 \rangle && (128)\\
&= \left[\frac{2j+1}{4\pi|c_j|}\right]^2 \int_{4\pi} d\Omega D^{j*}_{m\alpha}(\phi,\theta,-\phi) \int_{4\pi} d\Omega' D^{j}_{m\delta}(\phi',\theta',-\phi') \langle \theta', \phi', \lambda_1, \lambda_2 | \theta, \phi, \lambda_1, \lambda_2 \rangle \\
&= \left[\frac{2j+1}{4\pi|c_j|}\right]^2 \int_{4\pi} d\Omega D^{j*}_{m\alpha}(\phi,\theta,-\phi) \int_{4\pi} d\Omega' D^{j}_{m\alpha}(\phi',\theta',-\phi') \delta(\cos\theta' - \cos\theta)\delta(\phi' - \phi) \\
&= \left[\frac{2j+1}{4\pi|c_j|}\right]^2 \int_{4\pi} d\Omega D^{j*}_{m\alpha}(\phi,\theta,-\phi) D^{j}_{m\alpha}(\phi,\theta,-\phi) && (129)\\
&= 2\pi \left[\frac{2j+1}{4\pi|c_j|}\right]^2 \int_{-1}^{1} d\cos\theta \left[d^j_{m\alpha}(\theta)\right]^2 && (130)\\
&= \frac{4\pi}{2j+1}\left[\frac{2j+1}{4\pi|b_j|}\right]^2. && (131)
\end{aligned}
$$

Therefore, $|c_j|^2 = (2j+1)/4\pi$, or picking a phase convention,

$$
c_j = \sqrt{\frac{4\pi}{2j+1}}. \qquad (132)
$$

where we assume that it is all right to interchange the summation and integration. Since each term is non-negative (and each finite), there is no potential for cancellations. Hence, if we find convergence for one ordering of the operations, we will for the other as well.

Note that we have used the result of Eqn. 348 of my angular momentum notes to obtain:

$$
\int_{-1}^{1} d\cos\theta \left[d^j_{m\alpha}(\theta)\right]^2 = \frac{2}{2j+1}. \qquad (133)
$$

12. In the notes we derived the optical theorem assuming that we had a "symmetric central force". Show that this assumption is unnecessary. Hint: This is trivial, except for one piece of the assumption which you will have to retain.

    Solution: Start with the step prior to making the assumption in the notes:

$$
-\frac{i}{2\pi}\frac{\delta(p'-p'')}{p'}\left[f(\mathbf{p}',\mathbf{p}'') - f^*(\mathbf{p}'',\mathbf{p}')\right] = \frac{\delta(p'-p'')}{4\pi^2}\int_{(4\pi)} d\Omega_u f(\mathbf{p}',\mathbf{q}) f^*(\mathbf{p}'',\mathbf{q}). \qquad (134)
$$

Note that we must have $p' = p'' = q \equiv p$. Thus, write:

$$-\frac{i}{p}\left[f(p\mathbf{u}', p\mathbf{u}'') - f^*(p\mathbf{u}'', p\mathbf{u}')\right] = \frac{1}{2\pi}\int_{(4\pi)} d\Omega_u f(p\mathbf{u}', p\mathbf{u}) f^*(p\mathbf{u}'', p\mathbf{u}). \tag{135}$$

Now consider foward scattering: $\mathbf{u}'' = \mathbf{u}'$:

$$-\frac{i}{p}\left[f(p\mathbf{u}', p\mathbf{u}') - f^*(p\mathbf{u}', p\mathbf{u}')\right] = \frac{1}{2\pi}\int_{(4\pi)} d\Omega_u f(p\mathbf{u}', p\mathbf{u}) f^*(p\mathbf{u}', p\mathbf{u}). \tag{136}$$

With the assumption that $f(p\mathbf{u}', p\mathbf{u}) = f(p\mathbf{u}, p\mathbf{u}')$, we immediately see that we have once again the optical theorem:

$$\sigma_T(p) = \frac{4\pi}{p}\Im f(p;1). \tag{137}$$

Note that the assumption we retained was that the scattering amplitude is invariant (up to a phase) under interchange of incoming and outgoing directions.

References

[1] Eyvind H. Wichmann, "Scattering of Wave Packets", *American Journal of Physics*, **33** (1965) 20-31.

[2] M. Jacob and G. C. Wick, "On the General Theory of Collisions for Particles with Spin", *Annals of Physics*, **7** (1959) 404.