Physics 195
Course Notes
Second Quantization
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1 Introduction

This note is an introduction to the topic of "second quantization", and hence to quantum "field theory". In the Electromagnetic Interactions note, we have already been exposed to these ideas in our quantization of the electromagnetic field in terms of photons. We develop the concepts more generally here, for both bosons and fermions. One of the uses of this new formalism is that it provides a powerful structure for dealing with the symmetries of the states and operators for systems with many identical particles.

2 Creation and Annihilation Operators

We begin with the idea that emerged in our quantization of the electromagnetic field, and introduce operators that add or remove particles from a system, similar to the changing of excitation quanta of a harmonic oscillator.

To follow an explicit example, suppose that we have a potential well, $V(\mathbf{x})$, with single particle eigenstates $\phi_0(\mathbf{x}), \phi_1(\mathbf{x}), \ldots$ Suppose we have an n (identical) boson system, where all n bosons are in the lowest, ϕ_0, level. Denote this state by $|n\rangle$. We assume that $|n\rangle$ is normalized: $\langle n|n\rangle = 1$. Since the particles are bosons, we can have $n = 0, 1, 2, \ldots$, where $|0\rangle$ is the state with no particles (referred to as the "vacuum").

Now define "annihilation" (or "destruction") operators according to:

$$b_0|n\rangle = \sqrt{n}|n-1\rangle \quad (1)$$

$$b_0^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \quad (2)$$

Note that these operators subtract or add a particle to the system, in the state ϕ_0. They have been defined so that their algebraic properties are identical to the raising/owering operators of the harmonic oscillator. For example, consider the commutator:

$$[b_0, b_0^\dagger]|n\rangle = (b_0 b_0^\dagger - b_0^\dagger b_0)|n\rangle \quad (3)$$

$$= [(n+1) - (n)]\,|n\rangle \quad (4)$$

$$= |n\rangle. \quad (5)$$

Thus $[b_0, b_0^\dagger] = 1$. With these operators, we may write the n-particle state in terms of the vacuum state by:

$$|n\rangle = \frac{(b_0^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{6}$$

As in the case of the harmonic oscillator, $b_0^\dagger$ is the hermitian conjugate of b_0. To see this, consider the following: We have $b_0^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Thus,

$$\langle n+1|b_0^\dagger|n\rangle = \sqrt{n+1}, \tag{7}$$

and hence, $\langle n+1|b_0^\dagger = \sqrt{n+1}\langle n|$, or

$$\langle n|b_0^\dagger = \sqrt{n}\langle n-1|. \tag{8}$$

Likewise, b_0 acts as a creation operator when acting to the left:

$$\langle n|b_0 = \sqrt{n+1}\langle n+1|. \tag{9}$$

We may write the n-particle state in terms of the vacuum state by:

$$|n\rangle = \frac{(b_0^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{10}$$

Finally, we have the "number of particles" operator: $B_0 \equiv b_0^\dagger b_0$, with

$$B_0|n\rangle = n|n\rangle. \tag{11}$$

Now suppose that the particles are fermions, and define fermion annihilation and creation operators:

$$f_0|1\rangle = |0\rangle, \qquad f_0|0\rangle = 0; \tag{12}$$
$$f_0^\dagger|1\rangle = 0, \qquad f_0^\dagger|0\rangle = |1\rangle. \tag{13}$$

In the $|0\rangle, |1\rangle$ basis, these operators are the 2×2 matrices:

$$f_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f_0^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{14}$$

With this explicit representation, we see that they are hermitian conjugate to each other. By construction, we cannot put two fermions in the same state with these operators.

The algebraic properties of the fermion operators are different from those of the boson operators. The commutator, in the $|0\rangle, |1\rangle$ basis, is

$$[f_0, f_0^\dagger] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq I. \tag{15}$$

Consider the anticommutator:

$$\begin{aligned} \{f_0, f_0^\dagger\}|1\rangle &= (f_0 f_0^\dagger + f_0^\dagger f_0)|1\rangle = |1\rangle, && (16) \\ \{f_0, f_0^\dagger\}|0\rangle &= |0\rangle && (17) \end{aligned}$$

That is, $\{f_0, f_0^\dagger\} = 1$. Also,

$$\begin{aligned} \{f_0, f_0\} &= 0, && (18) \\ \{f_0^\dagger, f_0^\dagger\} &= 0. && (19) \end{aligned}$$

The number of particles operator is $F_0 = f_0^\dagger f_0$.

Now return to bosons, and consider two levels, ϕ_0 and ϕ_1. Let $|n_0, n_1\rangle$ be the state with n_0 bosons in ϕ_0 and n_1 bosons in ϕ_1. As before, define,

$$\begin{aligned} b_0|n_0, n_1\rangle &= \sqrt{n_0}|n_0 - 1, n_1\rangle && (20) \\ b_0^\dagger|n_0, n_1\rangle &= \sqrt{n_0 + 1}|n_0 + 1, n_1\rangle, && (21) \end{aligned}$$

and also,

$$\begin{aligned} b_1|n_0, n_1\rangle &= \sqrt{n_1}|n_0, n_1 - 1\rangle && (22) \\ b_1^\dagger|n_0, n_1\rangle &= \sqrt{n_1 + 1}|n_0, n_1 + 1\rangle. && (23) \end{aligned}$$

In addition to the earlier commutation relations, we have that the annihilation and creation operators for different levels commute with each other:

$$\begin{aligned} [b_0, b_1] &= 0; \qquad [b_0^\dagger, b_1] = 0 && (24) \\ \left[b_0, b_1^\dagger\right] &= 0; \qquad [b_0^\dagger, b_1^\dagger] = 0 && (25) \end{aligned}$$

We can construct an arbitrary state from the vacuum by:

$$|n_0, n_1\rangle = \frac{(b_0^\dagger)^{n_0}}{\sqrt{n_0!}} \frac{(b_1^\dagger)^{n_1}}{\sqrt{n_1!}} |0, 0\rangle \tag{26}$$

The total number operator is now

$$B = B_0 + B_1 = b_0^\dagger b_0 + b_1^\dagger b_1, \tag{27}$$

so that

$$B|n_0, n_1\rangle = (n_0 + n_1)|n_0, n_1\rangle. \tag{28}$$

In the case of fermions, we now have four possible states: $|0,0\rangle$, $|1,0\rangle$, $|0,1\rangle$, and $|1,1\rangle$. We define:

$$\begin{aligned}
f_0^\dagger|0,0\rangle &= |1,0\rangle; \quad f_0^\dagger|1,0\rangle = 0, & (29)\\
f_0|1,0\rangle &= |0,0\rangle; \quad f_0|0,0\rangle = 0, & (30)\\
f_0|0,1\rangle &= 0; \quad f_0^\dagger|1,1\rangle = 0, & (31)\\
f_1^\dagger|0,0\rangle &= |0,1\rangle; \quad f_1|0,0\rangle = f_1|1,0\rangle = 0, & (32)\\
f_1^\dagger|1,0\rangle &= |1,1\rangle; \quad f_1|0,1\rangle = |0,0\rangle, & (33)\\
f_1^\dagger|0,1\rangle &= f_1^\dagger|1,1\rangle = 0; \quad f_1|1,1\rangle = |1,0\rangle. & (34)
\end{aligned}$$

But we must be careful in writing down the remaining actions, of $f_0, f_0^\dagger$ on the states with $n_1 = 1$. These actions are constrained by consistency with the exclusion principle. We must get a sign change if we interchange the two fermions in a state. Thus, consider using the f and $f^\dagger$ operators to "interchange" the two fermions in the $|1,1\rangle$ state: First, take the fermion away from ϕ_1,

$$|1,1\rangle \to |1,0\rangle = f_1|1,1\rangle. \tag{35}$$

Then "move" the other fermion from ϕ_0 to ϕ_1,

$$|1,0\rangle \to |0,1\rangle = f_1^\dagger f_0|1,0\rangle. \tag{36}$$

Finally, restore the other one to ϕ_0,

$$|0,1\rangle \to f_0^\dagger|0,1\rangle = f_0^\dagger f_1^\dagger f_0 f_1|1,1\rangle \tag{37}$$

We require the result to be a sign change, *i.e.*,

$$f_0^\dagger|0,1\rangle = -|1,1\rangle. \tag{38}$$

Since f_0 is the hermitian conjugate of $f_0^\dagger$, we also have $f_0|1,1\rangle = -|0,1\rangle$.

We therefore have the anticommutation relations:

$$\{f_0, f_0^\dagger\} = \{f_1, f_1^\dagger\} = 1. \tag{39}$$

All other anticommutators are zero, including $\{f_0, f_1\} = \{f_0, f_1^\dagger\} = 0$, following from the antisymmetry of fermion states under interchange.

We may generalize these results to spaces with an arbitrary number of single particle states. Thus, let $|n_0, n_1, \ldots\rangle$ be a vector in such a space. For the case of bosons, we have, in general:

$$\begin{aligned} [b_i, b_j^\dagger] &= \delta_{ij}, & (40)\\ [b_i, b_j] &= [b_i^\dagger, b_j^\dagger] = 0, & (41)\\ |n_0, n_1, \ldots\rangle &= \cdots \frac{(b_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(b_0^\dagger)^{n_0}}{\sqrt{n_0!}} |0\rangle, & (42) \end{aligned}$$

where $|0\rangle$ represents the vacuum state, with all $n_i = 0$. Note that these are the same as the photon annihilation and creation operators $\hat{A}^\dagger, \hat{A}$, that we defined in the Electromagnetic Interactions note, except for the $\sqrt{2\pi/\omega}$ factor.

For the fermion case, we have the generalization:

$$\begin{aligned} \{f_i, f_j^\dagger\} &= \delta_{ij}, & (43)\\ \{f_i, f_j\} &= \{f_i^\dagger, f_j^\dagger\} = 0, & (44)\\ |n_0, n_1, \ldots\rangle &= \cdots (f_1^\dagger)^{n_1} (f_0^\dagger)^{n_0} |0\rangle. & (45) \end{aligned}$$

The number operators are similar in both cases:

$$\begin{aligned} B &= \sum B_i = \sum_i b_i^\dagger b_i, & (46)\\ F &= \sum F_i = \sum_i f_i^\dagger f_i, & (47) \end{aligned}$$

and $[B_i, B_j] = [F_i, F_j] = 0$.

3 Field Operators

Consider now plane wave states in a box (rectangular volume V, sides $L_i, i = 1, 2, 3$), with periodic boundary conditions:

$$\phi_{\mathbf{k}}(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}}, \tag{48}$$

where $k_i = 2\pi n_j/L_i$, $n_j = 0, \pm 1, \pm 2, \ldots$. The creation operator $a_{\mathbf{k}s}^\dagger$ (a is either b or f, for bosons or fermions, respectively), adds a particle with momentum $\mathbf{k}$ and spin projection s; the annilation operator $a_{\mathbf{k}s}$ removes one. Note that $\phi_{\mathbf{k}}(\mathbf{x})$ is the amplitude at $\mathbf{x}$ to find a particle added by $a^\dagger_{\mathbf{k}s}$.

Now consider the operator:

$$\psi^\dagger_s(\mathbf{x}) \equiv \sum_{\mathbf{k}} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} a^\dagger_{\mathbf{k}s}. \tag{49}$$

This operator adds a particle in a superpositon of momentum states with amplitude $\frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}}$, so that the amplitude for finding the particle at $\mathbf{x}'$ added by $\psi^\dagger_s(\mathbf{x})$ is a coherent sum of amplitudes $e^{i\mathbf{k}\cdot\mathbf{x}'}/\sqrt{V}$, with coefficients $e^{-i\mathbf{k}\cdot\mathbf{x}}/\sqrt{V}$. That is, the amplitude at $\mathbf{x}'$ is

$$\sum_{\mathbf{k}} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}'}}{\sqrt{V}} = \delta^{(3)}(\mathbf{x}-\mathbf{x}') \tag{50}$$

[by Fourier series expansion of $\delta^{(3)}(\mathbf{x}-\mathbf{x}')$:

$$g(\mathbf{x}') = \frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}'} \int_V d^3(\mathbf{x}'') e^{-i\mathbf{k}\cdot\mathbf{x}''} g(\mathbf{x}''), \tag{51}$$

with $g(\mathbf{x}') = \delta^{(3)}(\mathbf{x}-\mathbf{x}')$].

The operator $\psi^\dagger_s(\mathbf{x})$ thus adds a particle at $\mathbf{x}$ – it creates a particle at point $\mathbf{x}$ (with spin projection s). Likewise, the operator

$$\psi_s(\mathbf{x}) \equiv \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} a_{\mathbf{k}s} \tag{52}$$

removes a particle at $\mathbf{x}$. The operators $\psi^\dagger_s(\mathbf{x})$ and $\psi_s(\mathbf{x})$ are called "field operators". They have commutation relations following from the commutation relations for the a and $a^\dagger$ operators:

$$\psi_s(\mathbf{x})\psi_{s'}(\mathbf{x}') \pm \psi_{s'}(\mathbf{x}')\psi_s(\mathbf{x}) = 0 \tag{53}$$

$$\psi^\dagger_s(\mathbf{x})\psi^\dagger_{s'}(\mathbf{x}') \pm \psi^\dagger_{s'}(\mathbf{x}')\psi^\dagger_s(\mathbf{x}) = 0, \tag{54}$$

where the upper sign is for fermions, and the lower sign is for bosons. For bosons, adding (or removing) a particle at $\mathbf{x}$ commutes with adding one at $\mathbf{x}'$. For fermions, adding (or removing) a particle at $\mathbf{x}$ anticommutes with adding one at $\mathbf{x}'$. Also,

$$\psi_s(\mathbf{x})\psi^\dagger_{s'}(\mathbf{x}') \pm \psi^\dagger_{s'}(\mathbf{x}')\psi_s(\mathbf{x}) = \sum_{\mathbf{k},\mathbf{k}'} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}e^{i\mathbf{k}'\cdot\mathbf{x}'}}{V} \begin{cases} \{f_{\mathbf{k}s}, f^\dagger_{\mathbf{k}'s'}\} \\ [b_{\mathbf{k}s}, b^\dagger_{\mathbf{k}'s'}] \end{cases} \tag{55}$$

$$= \sum_{\mathbf{k},\mathbf{k}'} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}e^{i\mathbf{k}'\cdot\mathbf{x}'}}{V}\delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'} \tag{56}$$

$$= \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}'}}{V}\delta_{ss'} \tag{57}$$

$$= \delta(\mathbf{x}-\mathbf{x}')\delta_{ss'}. \tag{58}$$

Thus, adding particles commutes (bosons) or anticommutes (fermions) with removing them, unless it is at the same point and spin projection. If it is at the same point (and spin projection) we may consider the case with no particle originally there – the $\psi^\dagger\psi$ term gives zero, but the $\psi\psi^\dagger$ term does not, since it creates a particle which it then removes.

If we suppress the spin indices, we construct a state with n particles at $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n$ by:

$$|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n\rangle = \frac{1}{\sqrt{n!}}\psi^\dagger(\mathbf{x}_n)\ldots\psi^\dagger(\mathbf{x}_1)|0\rangle. \tag{59}$$

Note that such states form a useful basis for systems of many identical particles, since, by the commutation relations of the $\psi^\dagger$'s, they have the desired symmetry under interchanges of $\mathbf{x}_i$'s.[1] For example, for fermions,

$$\psi^\dagger(\mathbf{x}_2)\psi^\dagger(\mathbf{x}_1) = -\psi^\dagger(\mathbf{x}_1)\psi^\dagger(\mathbf{x}_2) \tag{60}$$

gives

$$|\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \ldots, \mathbf{x}_n\rangle = -|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n\rangle. \tag{61}$$

Note also that we can add another particle, and automatically maintain the desired symmetry:

$$\psi^\dagger(\mathbf{x})|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n\rangle = \sqrt{n+1}|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n, \mathbf{x}\rangle. \tag{62}$$

[1]These Hilbert spaces of multiple, variable numbers of particles, are known as Fock spaces.

Now let us evaluate:

$$\begin{aligned}\psi(\mathbf{x})|\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\ldots,\mathbf{x}_n\rangle &= \frac{1}{\sqrt{n!}}\psi(\mathbf{x})\psi^\dagger(\mathbf{x_n})\ldots\psi^\dagger(\mathbf{x}_1)|0\rangle \\ &= \frac{1}{\sqrt{n!}}\left[\delta^{(3)}(\mathbf{x}-\mathbf{x}_n)\pm\psi^\dagger(\mathbf{x}_n)\psi(\mathbf{x})\right]\psi^\dagger(\mathbf{x}_{n-1})\ldots\psi^\dagger(\mathbf{x}_1)|0\rangle \\ &= \frac{1}{\sqrt{n!}}\Big[\delta^{(3)}(\mathbf{x}-\mathbf{x}_n)|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_{n-1}\rangle \\ &\quad \pm\delta^{(3)}(\mathbf{x}-\mathbf{x}_{n-1})|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_{n-2},\mathbf{x}_n\rangle \\ &\quad +\ldots+ \\ &\quad (\pm)^{n-1}\delta^{(3)}(\mathbf{x}-\mathbf{x}_1)|\mathbf{x}_2,\mathbf{x}_2,\ldots,\mathbf{x}_n\rangle\Big], \end{aligned} \tag{63}$$

where the upper sign is for bosons and the lower for fermions. This quantity is non-zero if and only if $\mathbf{x}=\mathbf{x}_j$ (and the corresponding suppressed spin projections are also the same). If this is the case, the $n-1$ particle state which remains after performing the operation has the correct symmetry.

Note that

$$\begin{aligned}\langle\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n| &= \frac{1}{\sqrt{n!}}\left[\psi^\dagger(\mathbf{x}_n)\psi^\dagger(\mathbf{x}_{n-1})\ldots\psi^\dagger(\mathbf{x}_1)|0\rangle\right]^\dagger \\ &= \langle 0|\psi(\mathbf{x}_1)\ldots\psi(\mathbf{x}_n)\frac{1}{\sqrt{n!}}. \end{aligned} \tag{64}$$

Thus, by iterating the above repeated commutation process we calculate:

$$\langle\mathbf{x}'_1,\mathbf{x}'_2,\ldots,\mathbf{x}'_n|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\rangle = \delta_{nn'}\sum_P(\pm)^P P\left[\delta(\mathbf{x}_1-\mathbf{x}'_1)\delta(\mathbf{x}_2-\mathbf{x}'_2)\ldots\delta(\mathbf{x}_n-\mathbf{x}'_n)\right], \tag{65}$$

where $\sum_P$ is a sum over all permutations, P, of $\mathbf{x}'_1,\mathbf{x}'_2,\ldots,\mathbf{x}'_n$ and the $(-)^P$ factor for fermions inserts a minus sign for odd permutations.

Suppose we wish to create an n particle state $\phi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$ which has the desired symmetry, even if ϕ itself does not. The desired state is:

$$|\Phi\rangle = \int d^3(\mathbf{x}_1)\ldots d^3(\mathbf{x}_n)\phi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\rangle. \tag{66}$$

We can calculate the amplitude for observing the particles at $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n$ by:

$$\begin{aligned}\langle\mathbf{x}'_1,\mathbf{x}'_2,\ldots,\mathbf{x}'_n|\Phi\rangle &= \int d^3(\mathbf{x}_1)\ldots d^3(\mathbf{x}_n)\phi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\langle\mathbf{x}'_1,\mathbf{x}'_2,\ldots,\mathbf{x}'_n|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\rangle \\ &= \frac{1}{n!}\sum_P(\pm)^P P\phi(\mathbf{x}'_1,\mathbf{x}'_2,\ldots,\mathbf{x}'_n). \end{aligned} \tag{67}$$

That is, $\langle \mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n|\Phi\rangle$ is properly symmetrized. If ϕ is already properly symmetrized, then all $n!$ terms in $\sum_P$ are equal and $\langle \mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n|\Phi\rangle = \phi(\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n)$. If ϕ is normalized to one, and symmetrized, we have:

$$\begin{aligned}
\langle\Phi|\Phi\rangle &= \int d^3(\mathbf{x}_1)\ldots d^3(\mathbf{x}_n)\phi^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x_n})\langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x_n}| \\
&\quad \int d^3(\mathbf{x}'_1)\ldots d^3(\mathbf{x}'_n)\phi(\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n)|\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n\rangle \\
&= \int d^3(\mathbf{x}_1)\ldots d^3(\mathbf{x}_n)\int d^3(\mathbf{x}'_1)\ldots d^3(\mathbf{x}'_n) \\
&\quad \phi^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x_n})\phi(\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n)\frac{1}{n!}\sum_P(\pm)^P P\left[\delta(\mathbf{x}_1-\mathbf{x}'_1)\delta(\mathbf{x}_2-\mathbf{x}'_2)\ldots\delta(\mathbf{x}_n-\mathbf{x}'_n)\right] \\
&= \int d^3(\mathbf{x}_1)\ldots d^3(\mathbf{x}_n)|\phi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x_n})|^2 \qquad (68) \\
&= 1. \qquad (69)
\end{aligned}$$

We may write the state $|\Phi\rangle$ in terms of an expansion in the amplitudes $\langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n|\Phi\rangle$ for observing the particles at $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$:

$$|\Phi\rangle = \int d^3(\mathbf{x}_1)\ldots d^3(\mathbf{x}_n)|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\rangle\langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n|\Phi\rangle. \qquad (70)$$

That is, we have the identity operator on symmetrized n particle states:

$$I_n = \int d^3(\mathbf{x}_1)\ldots d^3(\mathbf{x}_n)|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\rangle\langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n|. \qquad (71)$$

If $|\Phi\rangle$ is an n particle state, then

$$I_{n'}|\Phi\rangle = \delta_{nn'}|\Phi\rangle. \qquad (72)$$

Summing the n particle identity operators gives the identity on the symmetrized states of any number of particles: $I = \sum_{n=0}^{\infty} I_n$, where $I_0 = |0\rangle\langle 0|$.

4 Exercises

1. Consider a two-level fermion system. With respect to basis $|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle$, construct the explisict 4×4 matrices representing the creation and annihilation operators $f_0, f_1, f_0^\dagger, f_1^\dagger$. Check that the desired anticommutation relations are satisfied. Form the explicit matrix representation of the total number operator.

2. You showed in Exercise 1 of the Electromagnetic Interactions course note that under a gauge transformation:

$$\mathbf{A}(\mathbf{x},t) \rightarrow \mathbf{A}'(\mathbf{x},t) = \mathbf{A}(\mathbf{x},t) + \nabla\chi(\mathbf{x},t) \tag{73}$$
$$\Phi(\mathbf{x},t) \rightarrow \Phi'(\mathbf{x},t) = \Phi(\mathbf{x},t) - \partial_t\chi(\mathbf{x},t), \tag{74}$$

that the wave function (the solution to the Schrödinger equation) has the corresponding transformation:

$$\psi'(\mathbf{x},t) = e^{iq\chi(\mathbf{x},t)}\psi(\mathbf{x},t). \tag{75}$$

Generalize this result to the case of an N particle system.