

Now that we have the Hamiltonian we can return to the problem of the order parameter.

Recall from our discussion of density matrices that the Helmholtz free energy,

$$F = -T \ln Z \quad (k_B = 1)$$

where $Z(T)$ is the partition function: $Z(T) = \text{Tr}(e^{-H/T})$

Or,

$$F = -T \ln[\text{Tr}(e^{-H/T})]$$

Now $H = -H_c + H_s$. But H_c is just a number, hence

$$\begin{aligned} F &= -T \ln[\text{Tr}(e^{-(H_s - H_c)/T})] \\ &= -T \ln[\text{Tr}(e^{H_c/T} e^{-H_s/T})] \\ &= -T \ln[e^{H_c/T} \text{Tr}(e^{-H_s/T})] \\ &= -T \ln e^{H_c/T} - T \ln[\text{Tr} e^{-H_s/T}] \\ &= -H_c - T \ln[\text{Tr} e^{-H_s/T}] \end{aligned}$$

Now

$$-H_c = -\frac{1}{2} \int dx \int dy \Phi_{ss'}(x, y) U(x-y) \Phi_{ss'}^*(x, y)$$

Hence

$$\begin{aligned} F &= -\frac{1}{2} \int dx \int dy \Phi_{ss'}(x, y) U(x-y) \Phi_{ss'}^*(x, y) \\ &\quad - T \ln[\text{Tr} e^{-H_s/T}] \end{aligned}$$

Assume that our system is in thermal contact with a heat reservoir, and that it is in equilibrium. Since it is a piece of metal, it has constant volume.

Thus, the free energy must be at a minimum, with respect to variations in the order parameter:

$$\frac{\delta F}{\delta \Phi_{ss'}^*(x, y)} = 0$$

$$= -\frac{1}{2} \Phi_{ss'}(x, y) U(x-y)$$

$$- T \frac{1}{\text{Tr} [e^{-H_s/T}]} \frac{\delta}{\delta \Phi_{ss'}^*(x, y)} \text{Tr} e^{-H_s/T}$$

$$= -\frac{1}{2} \Phi_{ss'}(x, y) U(x-y) + \frac{\text{Tr} [e^{-H_s/T} \frac{\delta H_s}{\delta \Phi_{ss'}^*(x, y)}]}{\text{Tr} [e^{-H_s/T}]}$$

[Note that a) Φ, Φ^* are independent

$$b) \frac{\delta}{\delta g(y)} \int f(x') g(x') dx' = f(x)$$

$$= \int dx' f(x') \frac{\delta g(x')}{\delta g(y)}$$

Need

$$\frac{\delta H_s}{\delta \Phi_{ss'}^*(x, y)} = \frac{1}{2} \psi_s^+(x) \psi_{s'}^+(y) U(x-y)$$

Hence,

$$\Phi_{ss'}(x, y) = \frac{\text{Tr} [e^{-H_s/T} \psi_s^+(x) \psi_{s'}^+(y)]}{\text{Tr} [e^{-H_s/T}]}$$

Recall from our density matrix note, that in the canonical ensemble:

$$\langle Q \rangle = \text{Tr} [Q \rho(T)] = \text{Tr} [Q \frac{e^{-H/T}}{Z(T)}]$$

$$= \frac{\text{Tr} [Q e^{-H/T}]}{\text{Tr} [e^{-H/T}]}$$

Thus, our expression for $\Phi_{SS'}(x, y)$ is consistent with our definition:

$$\Phi_{SS'}(x, y) = \langle \psi_S^\dagger(x) \psi_{S'}^\dagger(y) \rangle$$

Plus, we have a self-consistency condition in that H_S is given in terms of $\Phi_{SS'}(x, y)$.

Energy Gap

We remark on a couple of experimental observations on typical superconductors:

1) At DC and low frequencies, zero resistance. "insolent"

At optical frequencies, same behavior as normal metals (appearance doesn't change).

"Appearance" starts to change at \sim few $\times 10^{14}$ Hz

→ Below this frequency, superconducting

Above " " " , normal resistance.

2) The ^{electronic} specific heat of a superconductor, ^{well} below T_c depends exponentially on temperature:

$$[C_{el}]_{\text{super}} \sim e^{-\Delta/T}$$

Both observations are suggestive of an energy gap.

e.g., ~~we~~ excitations below a certain energy (frequency) are insufficient to remove electrons from a "superconducting" bands. Above the energy, the excitations are sufficient to cause electrons to cross the gap.

The energy gap is a key aspect of BCS theory.
Define the "gap function"

$$\Delta_{ss'}(x, y) \equiv \langle \psi_s^\dagger(x) \psi_{s'}^\dagger(y) \rangle U(x-y)$$

We get

$$\Delta_{ss'}^*(x, y) = \langle \psi_{s'}(y) \psi_s(x) \rangle U(x-y)$$

We may rewrite the Hamiltonian in terms of this gap function:

$$\begin{aligned} H_S &= \int dx \left(\frac{1}{2m} \nabla \psi_s^\dagger \cdot \nabla \psi_s - \mu \psi_s^\dagger \psi_s \right) \\ &+ \frac{1}{2} \int dx \int dy \left[\Delta_{ss'}(x, y) U(x-y) \psi_{s'}(y) \psi_s(x) \right. \\ &\quad \left. + \psi_s^\dagger(x) \psi_{s'}^\dagger(y) U(x-y) \Delta_{ss'}^*(x, y) \right] \\ &= \int dx \left(\frac{1}{2m} \nabla \psi_s^\dagger \cdot \nabla \psi_s - \mu \psi_s^\dagger \psi_s \right) \\ &+ \frac{1}{2} \int dx \int dy \left[\Delta_{ss'}(x, y) \psi_{s'}(y) \psi_s(x) \right. \\ &\quad \left. + \Delta_{ss'}^*(x, y) \psi_s^\dagger(x) \psi_{s'}^\dagger(y) \right] \end{aligned}$$

Our goal is to diagonalize the Hamiltonian, so that we may determine its spectrum.

Since H_S is bilinear in $\psi^{(\dagger)}$, we may diagonalize it, similarly to the SHO, via a linear transformation.

This linear transformation is known as the "Bogoliubov transformation".

First, rewrite H_S by defining a substantial shorthand:

$$\gamma_\alpha = \begin{pmatrix} \psi_s(x) \\ \psi_s^\dagger(x) \end{pmatrix} \quad \text{where } \alpha = (s, x)$$

With this new "2-component field vector" we have

$$\begin{aligned}
 \gamma_\alpha^\dagger \gamma_\alpha &= \int dx (\psi_s^\dagger(x), \psi_s(x)) \begin{pmatrix} \psi_s(x) \\ \psi_s^\dagger(x) \end{pmatrix} \\
 &= \int dx \{ \psi_s^\dagger(x), \psi_s(x) \} \\
 &= 1
 \end{aligned}$$

repeated
in text
⇒ $\int dx \sum_s$

Let us verify that we may write:

$$H_S = \frac{1}{2} \gamma_\alpha^\dagger H_{\alpha\beta} \gamma_\beta + H_0$$

where

$$H_0 \equiv \frac{1}{2} \text{Tr} \left[-\frac{1}{2m} \nabla^2 - \mu \right]$$

$$H_{\alpha\beta} = \begin{bmatrix} \left(\frac{1}{2m} \nabla_x \cdot \nabla_y - \mu \right) \delta_{\alpha\beta}(x-y) & \Delta_{\alpha\beta}^*(x,y) \\ -\Delta_{\alpha\beta}(x,y) & -\left(\frac{1}{2m} \nabla_x \cdot \nabla_y - \mu \right) \delta_{\alpha\beta}(x-y) \end{bmatrix}$$

[Exercise]

Note that the gap function is antisymmetric:

$$\begin{aligned}
 \Delta_{s's'}(y,x) &= \langle \psi_{s'}^\dagger(y) | \psi_s^\dagger(x) \rangle U(y-x) \\
 &= \langle -\psi_s^\dagger(x) | \psi_{s'}^\dagger(y) \rangle U(x-y) \quad \downarrow \text{assume} \\
 &= -\Delta_{s's'}(x,y)
 \end{aligned}$$

Hence

$$H_{\alpha\beta}^\dagger = H_{\alpha\beta} \quad H_0 \text{ is Hermitian}$$

Note that H_0 comes from the anticommutation relations. The trace is, of course, to be evaluated by taking the expectation value of the operator in any complete O.N. basis.

H_0 is, in fact, but the Δ will be canceled by another term.

We diagonalize $H_{\alpha\beta}$ by constructing its eigenvectors:

$$\varphi_{\alpha}^{(n)} = \begin{pmatrix} U_{\alpha}^{(n)} \\ V_{\alpha}^{(n)} \end{pmatrix},$$

where $n = \begin{pmatrix} + \\ - \end{pmatrix}$, labelling the two eigenvectors, s, k ← wave vector is transverse to vector

Thus,
$$H_{\alpha\beta} \varphi_{\beta}^{(n)} = \lambda_n \varphi_{\alpha}^{(n)}$$

Let λ be an eigenvalue of $H_{\alpha\beta}$

Consider $\Delta=0$; ("normal", not "superconductor")

In this case
$$H_{\alpha\beta} = \begin{bmatrix} (\frac{1}{2m} \nabla_x \cdot \nabla_y - \mu) \delta_{\alpha\beta}(x-y) & \\ 0 & -(\frac{1}{2m} \nabla_x \cdot \nabla_y - \mu) \delta_{\alpha\beta}(x-y) \end{bmatrix}$$

Our eigenvalue equation separates, and we may take plane wave solutions:

$$\varphi_{\alpha}^{(+,s)}(x; k) = \begin{pmatrix} U_{\alpha}^s(k) \\ 0 \end{pmatrix} e^{ikx}$$

$$\varphi_{\alpha}^{(-,s)}(x; k) = \begin{pmatrix} 0 \\ V_{\alpha}^s(k) \end{pmatrix} e^{ikx}$$

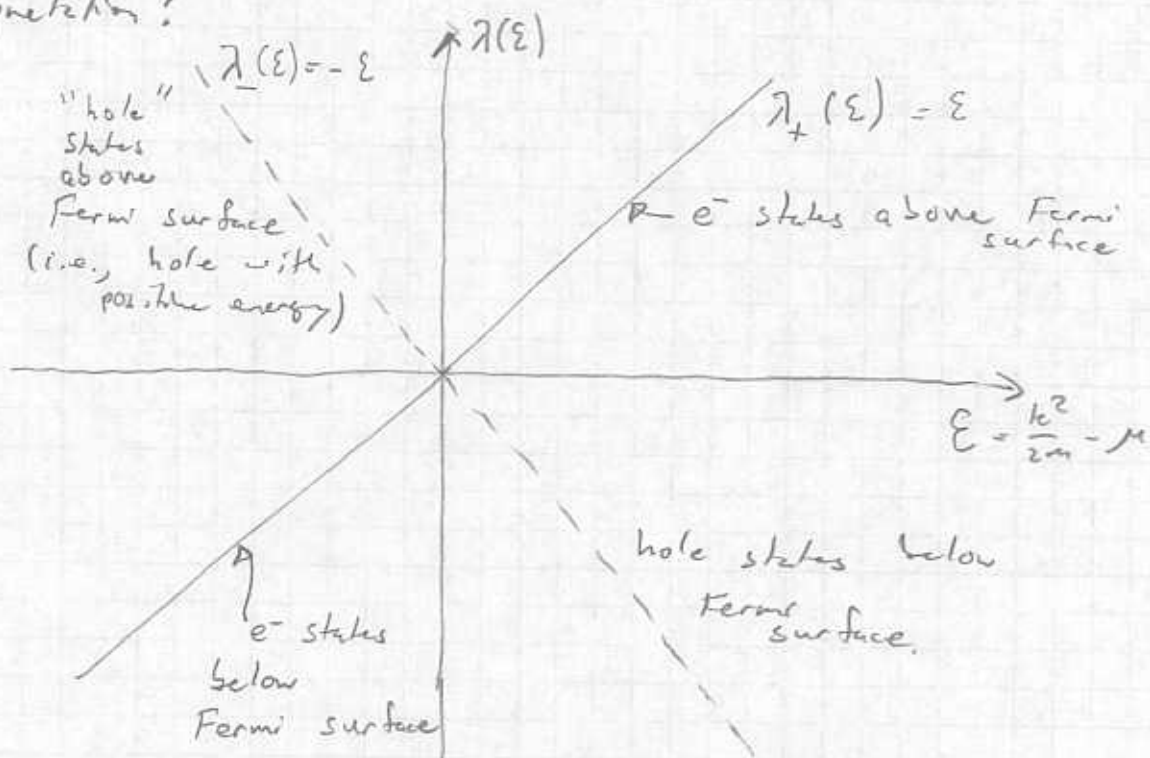
Notice that $\text{Tr}(H_{\alpha\beta}) = 0$. Thus, if $\lambda_+(k)$ is one eigenvalue, $\lambda_-(k) = -\lambda_+(k)$ must be the other eigenvalue.

We expect that $\lambda_+(k) = \frac{k^2}{2m} - \mu$

$$\lambda_-(k) = -\left(\frac{k^2}{2m} - \mu\right)$$

Let $\epsilon \equiv \frac{k^2}{2m} - \mu$ = energy relative to Fermi surface.
(for $\Delta = 0$)
($= \mu$ @ $T=0$)

As a function of ϵ , we have two branches, $\lambda_{\pm}(\epsilon)$, to our eigenvalue solutions. Let's try a "physical" interpretation:



Note that in the ground state, all ^{electron} negative energy levels are occupied, and all electron positive energy states are unoccupied. In terms of the hole states, all hole states above the Fermi surface

are occupied, and those below unoccupied.
 A creation operator on the "-" branch is an electron annihilation operator - creating a "negative energy" electron removes a positive energy electron.

We may create an excited state by removing an electron from below the Fermi surface - This creates a hole with positive energy.

Removes
 negative energy e^-
 creates a
 positive energy hole

Notice that, at $E=0$, there is a degeneracy - the two branches converge at the Fermi surface.

We had an example already in degenerate state P.T.;
 If we add a potential which removes the degeneracy, the states near the degeneracy are modified. A gap appears, in this case separating the positive and negative branches of the energy spectrum.
 This gap is a key feature of BCS superconductivity.

Let's make sure we understand our orthonormality and completeness, in terms of our eigenvectors of H_{sp} .

We choose Orthonormality \rightarrow

$$\sum_{\alpha} \phi_{\alpha}^{(n)*} \phi_{\alpha}^{(n')} = \delta_{nn'}$$

or, with

$$\phi_L^{(n)} = \begin{pmatrix} u_{\alpha}^{(n)} \\ v_{\alpha}^{(n)} \end{pmatrix},$$

Our eigenvectors of H_{df} are orthonormal and complete:

Orthonormality:

$$\sum_{\alpha} \varphi_{\alpha}^{(n)*} \varphi_{\alpha}^{(n')} = \delta_{nn'}$$

Completeness:

$$\sum_n \varphi_{\alpha}^{(n)*} \varphi_{\beta}^{(n)} = \delta_{\alpha\beta}$$

Finally, we complete our transformation: Express the original creation/annihilation fermion operators, γ_{α} , as a linear superposition:

$$\gamma_{\alpha} = \sum_n \varphi_{\alpha}^{(n)} A_n$$

where

A_n are the new "quasi-particle" operators

This is called a "Bogoliubov transformation".

The inverse transformation is:

[mult by $\varphi_{\alpha}^{(m)*}$ and \sum_{α}]:

$$\sum_{\alpha} \varphi_{\alpha}^{(m)*} \gamma_{\alpha} = \sum_n A_n \underbrace{\sum_{\alpha} \varphi_{\alpha}^{(m)*} \varphi_{\alpha}^{(n)}}_{\delta_{mn}}$$

$$= A_m$$

$$A_m = \begin{pmatrix} A_m^+ \\ A_m^- \end{pmatrix}$$

In expanded form, this transformation is:

"plus" component $A_n^+ = \int dx \sum_{\sigma} \left[u_{\sigma}^{s+}(x; k) \psi_{\sigma}(x) + v_{\sigma}^{s+}(x; k) \psi_{\sigma}^{\dagger}(x) \right]$
and $e^{(s; k)}$

"minus" component $A_n^- = \int dx \sum_{\sigma} \left[v_{\sigma}^s(x; k) \psi_{\sigma}(x) + u_{\sigma}^s(x; k) \psi_{\sigma}^{\dagger}(x) \right]$

A_n^- is the hermitian conjugate of A_n^+

To simplify, we write;

$$a_{\mu}(k) = A_n^+ \quad n = (\mu, k)$$

$$a_{\mu}^{\dagger}(k) = A_n^-$$

These quasiparticles satisfy the canonical anticommutation relations for fermions:

$$\{a_{\mu}(k), a_{\nu}^{\dagger}(k')\} = \delta_{\mu\nu} \delta_{kk'}$$

since,

$$\begin{aligned} \{a_{\mu}(k), a_{\nu}^{\dagger}(k')\} &= \int dx dy \sum_{\sigma, \sigma'} \left\{ u_{\sigma}^{\mu*}(x; k) \psi_{\sigma}(x) + v_{\sigma}^{\mu*}(x; k) \psi_{\sigma}^{\dagger}(x), \right. \\ &\quad \left. v_{\sigma'}^{\nu}(y; k') \psi_{\sigma'}(y) + u_{\sigma'}^{\nu}(y; k') \psi_{\sigma'}^{\dagger}(y) \right\} \\ &= \int dx dy \sum_{\sigma, \sigma'} \left[u_{\sigma}^{\mu*}(x; k) u_{\sigma'}^{\nu}(y; k') + v_{\sigma}^{\mu*}(x; k) v_{\sigma'}^{\nu}(y; k') \right] \\ &\quad \frac{\delta_{\sigma\sigma'} \delta(x-y)}{=} \{ \psi_{\sigma}(x), \psi_{\sigma'}^{\dagger}(y) \} \\ &= \int dx \sum_{\sigma} \left[u_{\sigma}^{\mu*}(x; k) u_{\sigma}^{\nu}(x; k') + v_{\sigma}^{\mu*}(x; k) v_{\sigma}^{\nu}(x; k') \right] \\ &= \delta_{\mu\nu} \delta_{kk'} \quad \text{from orthonormality condition} \\ &\quad \text{of } \varphi_{\alpha}^{(n)} \text{ vectors.} \end{aligned}$$

In terms of the u, v components, the Bogoliubov transformation is:

$$\psi_{\sigma}(x) = \sum_{k, \mu} \left[u_{\sigma}^{\mu}(x; k) a_{\mu}(k) + v_{\sigma}^{\mu*}(x; k) a_{\mu}^{\dagger}(k) \right]$$

$$\psi_{\sigma}^{\dagger}(x) = \sum_{k, \mu} \left[u_{\sigma}^{\mu*}(x; k) a_{\mu}^{\dagger}(k) + v_{\sigma}^{\mu}(x; k) a_{\mu}(k) \right]$$

↑
↑
↑

electron field operators
quasiparticle operators

In terms of the quasiparticle operators, the Hamiltonian is:

$$\begin{aligned}
 H_S (-H_0) &= \sum_{\alpha, \beta} \frac{1}{2} \gamma_{\alpha}^{\dagger} H_{\alpha\beta} \gamma_{\beta} \\
 &= \frac{1}{2} \sum_{\alpha, \beta, n} \sum_m \varphi_{\alpha}^{(n)\dagger} A_n^{\dagger} H_{\alpha\beta} \varphi_{\beta}^{(m)} A_m \\
 &= \frac{1}{2} \sum_{n, m} \underbrace{\sum_{\alpha} \varphi_{\alpha}^{(n)\dagger} A_n^{\dagger}}_{\delta_{nm}} \underbrace{\lambda_m \varphi_{\alpha}^{(m)}}_{A_m} A_m
 \end{aligned}$$

$$= \frac{1}{2} \sum_n \lambda_n A_n^{\dagger} A_n$$

$$\begin{aligned}
 A_n^{\dagger} &= \begin{pmatrix} A_n^{\dagger} \\ A_n^- \end{pmatrix}^{\dagger} = (A_n^-, A_n^{\dagger}) \\
 \lambda_n^{\dagger} &= -\lambda_n^-
 \end{aligned}$$

We may expand this sum and write the Hamiltonian in terms of the quasiparticle operators:

$$\begin{aligned}
 H_S &= \frac{1}{2} \sum_{\mu, k} \lambda_{\mu}(k) [a_{\mu}^{\dagger}(k) a_{\mu}(k) - a_{\mu}(k) a_{\mu}^{\dagger}(k)] + H_0 \\
 &= \sum_{\mu, k} \lambda_{\mu}(k) a_{\mu}^{\dagger}(k) a_{\mu}(k) - \frac{1}{2} \sum_{\mu, k} \lambda_{\mu}(k) + \frac{1}{2} \text{Tr} \left(-\frac{1}{2m} \nabla^2 - \mu \right)
 \end{aligned}$$

We have rewritten our Hamiltonian in the form of a Hamiltonian for non-interacting fermions. We have the number operator:

$$n_{\mu}(k) = a_{\mu}^{\dagger}(k) a_{\mu}(k)$$

with eigenvalues of 0 and 1 as usual.

Thus (up to an additive constant) the partition function is:

$$\begin{aligned}
 Z &= \text{Tr} e^{-H_S/T} \\
 &= \text{Tr} \left[e^{-\frac{1}{2T} \sum_{\mu, k} \lambda_{\mu}(k)} e^{-\frac{1}{2T} \text{Tr} \left(-\frac{\nabla^2}{2m} - \mu \right)} e^{\frac{1}{T} \sum_{\mu, k} \lambda_{\mu}(k) n_{\mu}(k)} \right] \\
 &= Z_0 \text{Tr} e^{-\frac{1}{T} \sum_{\mu, k} \lambda_{\mu}(k) n_{\mu}(k)}
 \end{aligned}$$

where

$$Z_0 = \exp \left\{ -\frac{1}{2T} \left[\text{Tr} \left(-\frac{\mathcal{D}^2}{2m} - \mu \right) - \sum_{\mu, k} \lambda_{\mu}(k) \right] \right\}$$

Hence

$$\begin{aligned} Z(T) &= Z_0(T) \text{Tr} \prod_{\mu, k} e^{-\frac{1}{T} \lambda_{\mu}(k) n_{\mu}(k)} \\ &= Z_0(T) \text{Tr} \prod_{\mu, k} \left[1 + e^{-\lambda_{\mu}(k)/T} \right] \end{aligned}$$

$\swarrow n_{\mu}(k)=0$ $\nwarrow n_{\mu}(k)=1$

Note that, while our form is that of non-interacting fermions, the energies $\lambda_{\mu}(k)$ depend on the gap function Δ via f and u, v .

We may evaluate the order parameter now in terms of averages on the quasiparticle fields

$$\begin{aligned} \Phi_{\alpha\beta}(x, y) &= \langle \psi_{\alpha}^{\dagger}(y) \psi_{\beta}^{\dagger}(x) \rangle \\ &= \sum_{k, \mu} \sum_{k', \nu} \langle [u_{\alpha}^{\mu*}(x, k) a_{\mu}^{\dagger}(k) + v_{\alpha}^{\mu}(x, k) a_{\mu}(k)] \\ &\quad [u_{\beta}^{\nu*}(y, k') a_{\nu}^{\dagger}(k') + v_{\beta}^{\nu}(y, k') a_{\nu}(k')] \rangle \end{aligned}$$

Thermal avg. \rightarrow

$$\begin{aligned} &= \sum_{k, \mu} \left\{ u_{\alpha}^{\mu*}(x, k) v_{\beta}^{\mu}(y, k) n(\mu, k) \right. \\ &\quad \left. + v_{\alpha}^{\mu}(x, k) u_{\beta}^{\mu*}(y, k) [1 - n(\mu, k)] \right\} \end{aligned}$$

where

$$n(\mu, k) = \frac{1}{1 + \exp(\lambda_{\mu}(k)/T)}$$

is the Fermi-Dirac distribution.

Once again, our equ. for $\Phi_{\alpha\beta}(x, y)$ is a self-consistent one, since u, v, λ depend on Φ via Δ . The key question to address is: Is there a non-trivial ($\Delta \neq 0$) self-consistent solution?

We are actually almost at the point where we can embark on performing this self-consistent analysis. But we won't quite make it. Instead (or, as the penultimate step), I'd like to develop our intuition a little further, without actually solving the problem.

Let's simplify a bit, and select a spin state.

Our order parameter can be expressed as spin-0 and spin-1 pieces. Let us consider the singlet piece: $S=0$.

The $S=0$ state is antisymmetric in the spin indices, hence symmetric in the space coordinates.

$$\begin{aligned} \text{Hence, } \Phi_{\alpha\beta}(x,y) &= -\Phi_{\beta\alpha}(x,y) \\ \Phi_{\alpha\beta}(x,y) &= \Phi_{\alpha\beta}(y,x) \end{aligned}$$

Consider translational invariance (assumed):

$T(a)$ = translation operator:

$$T(a) \psi_s(x) T^{-1}(a) = \psi_s(x+a)$$

Invariance \Rightarrow

$$\begin{aligned} \Phi_{\alpha\beta}(x,y) &= \langle \psi_{\alpha}^{+}(x) \psi_{\beta}^{+}(y) \rangle \\ &= \langle T(a) \psi_{\alpha}^{+}(x) T^{-1}(a) T(a) \psi_{\beta}^{+}(y) T^{-1}(a) \rangle \\ &= \langle \psi_{\alpha}^{+}(x+a) \psi_{\beta}^{+}(y+a) \rangle \\ &= \Phi_{\alpha\beta}(x+a, y+a) \end{aligned}$$

a arbitrary, e.g., if $a = -y$, then:

$$\Phi_{\alpha\beta}(x,y) = \Phi_{\alpha\beta}(x-y)$$

Assuming homogeneity & isotropy, convenient to Fourier transform:

$$\Phi_{\alpha\beta}(x-y) = \frac{1}{V} \sum_k e^{ik(x-y)} \Phi_{\alpha\beta}(k)$$

also

$$U(x-y) = \frac{1}{V} \sum_k e^{ik(x-y)} U(k)$$

and

$$\Delta_{\alpha\beta}(x-y) = \frac{1}{V} \sum_k e^{ik(x-y)} \Delta_{\alpha\beta}(k)$$

we have the convolution:

$$\Delta(k) = \frac{1}{V} \sum_{k'} U(k') \Phi(k-k')$$

$H_{\alpha\beta}$ translationally invariant \rightarrow plane wave eigenvectors \Rightarrow

$$u_{\alpha}^{\mu}(x; k) = \frac{1}{\sqrt{V}} u_{\alpha}^{\mu}(k) e^{ik \cdot x}$$

$$v_{\alpha}^{\mu}(x; k) = \frac{1}{\sqrt{V}} v_{\alpha}^{\mu}(k) e^{ik \cdot x}$$

substitute into

$$H_{\alpha\beta} \phi_{\beta}^{\mu} = \lambda_{\mu} \phi_{\alpha}^{\mu}$$

$$\textcircled{A} \quad \left(\frac{k^2}{2m} - \mu \right) u_{\alpha}^{\mu}(k) + \Delta_{\alpha\beta}^{\mu}(k) v_{\beta}^{\mu}(k) = \lambda_{\mu}(k) u_{\alpha}^{\mu}(k)$$

$$\textcircled{B} \quad \left(\frac{k^2}{2m} - \mu \right) v_{\alpha}^{\mu}(k) + \Delta_{\alpha\beta}^{\mu}(k) u_{\beta}^{\mu}(k) = -\lambda_{\mu}(k) v_{\alpha}^{\mu}(k)$$

$$\rightarrow \text{or: } \left[\frac{k^2}{2m} - \mu + \lambda_{\mu}(k) \right] v_{\alpha}^{\mu}(k) - \Delta_{\alpha\beta}^{\mu}(k) u_{\beta}^{\mu}(k)$$

mult. \textcircled{A} by $\left[\frac{k^2}{2m} - \mu + \lambda_{\mu}(k) \right] \Rightarrow$

$$\left[\frac{k^2}{2m} - \mu + \lambda_{\mu}(k) \right] \left[\frac{k^2}{2m} - \mu - \lambda_{\mu}(k) \right] u_{\alpha}^{\mu}(k) - \Delta_{\alpha\beta}^{\mu}(k) \Delta_{\beta\gamma}^{\mu}(k) u_{\gamma}^{\mu}(k) = 0$$

Now

$$\Delta_{d\beta}^* (k) \Delta_{\beta\gamma} (k) = \underbrace{\epsilon_{d\beta} \epsilon_{\beta\gamma}}_{\epsilon \text{ over spin index}} |\Delta(k)|^2$$

$$\epsilon_{d\beta} = -\epsilon_{\beta\gamma}$$

$$\epsilon_{+-} = 1$$

$$= -\delta_{\alpha\gamma} |\Delta(k)|^2$$

Also, letting

$$\epsilon = \frac{1}{2m} \hbar^2 k^2 - \mu$$

we have,

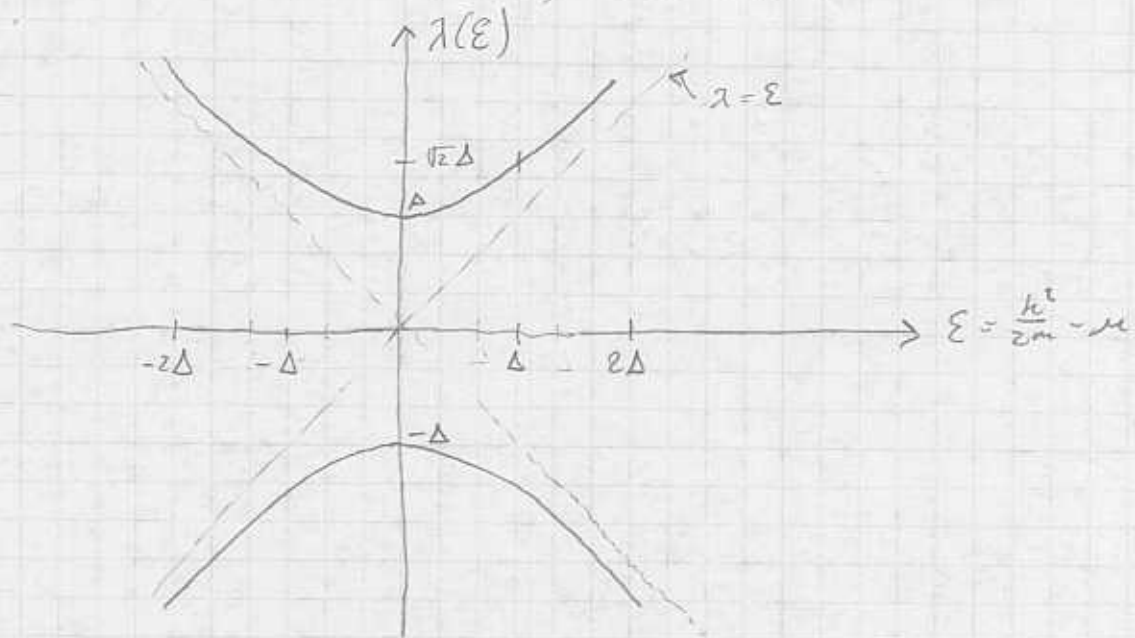
$$[\epsilon + \lambda(k)] [\epsilon - \lambda(k)] + |\Delta(k)|^2 = 0$$

$$\lambda^2 = \epsilon^2 + |\Delta|^2$$

or

$$\lambda(k) = \pm \sqrt{\epsilon^2(k) + |\Delta(k)|^2}$$

Thus, our quasiparticle energy spectrum, in the case $\Delta(k) = \Delta$ (independent of k), looks like:



In the superconducting ground state, all of the negative energy states are occupied, and the positive energy states are empty.

Near the Fermi surface, the quasiparticle states are superpositions of electron and hole states. Far away, they become electron-like and hole-like states again.

Note that there are both negative energy quasiparticles and positive energy quasiparticles. In the case $\Delta = 0$, the positive energy quasiparticles are ^{ordinary} electrons above the Fermi surface, and negative energy quasiparticles are ^{ordinary} electrons below the Fermi surface.

If $\Delta \neq 0$, this is altered in a region of order Δ about the Fermi surface.

In the superconducting state, there is a gap of size 2Δ separating the "top" of the ground state from the positive energy bands. The behavior of the ground state under application of external fields gives the superconductor its electromagnetic properties.