Magnetostatics I

By definition, in magnetostatics, no fields, including electric ones, are varying in time. However, magnetism is caused by moving electrical charges. We define a field, the electric current density, as

$$\vec{j}(\vec{r}) = |\rho^+(\vec{r})|\, \vec{v}^+(\vec{r}) - |\rho^-(\vec{r})|\, \vec{v}^-(\vec{r}),$$

where $\rho^+(\vec{r})$, $\vec{v}^+(\vec{r})$ are the density and average velocity of positive charges at $\vec{r}$ and $\rho^-(\vec{r})$, $\vec{v}^-(\vec{r})$ are similarly for the negative charges. The statics condition means that there is no time dependence in the current density. The charges in a differential volume $d^3 r$ at $\vec{r}$ may be flying about in all directions, but in the current density we need the average velocities. If it is ever necessary, we will assume that $|\rho^+(\vec{r})| = |\rho^-(\vec{r})|$ so that the charge density is zero

$$\rho(\vec{r}) = |\rho^+(\vec{r})| - |\rho^-(\vec{r})| = 0$$

and there are no electrostatic effects. This is an idealization for all conductors except superconducting ones because they need an electric field inside to drive the current (Ohm's law). We will use it when necessary, but usually, I will just assume a current and ignore the details of their generation. However, Ohm's law will enter the subject later.

Notice that the SI dimensions of the current density are Coulomb / meter^2 / second.

A fundamental law of great importance in the phenomenology of magnetostatics (and all of electromagnetism in general) is *charge conservation.* You can move charges around but you cannot create or destroy a net amount of charge. It is, however, to create and destroy exactly equal amounts of positive and negative charge. This happens in particle-antiparticle creation or annihilation, but we won't deal with such interesting things.

Consider charges in an infinitesimal volume $d^3 r = dx\, dy\, dz$. In a time dt the net amount of charge leaving the volume must equal the net decrease in the amount of charge in it.

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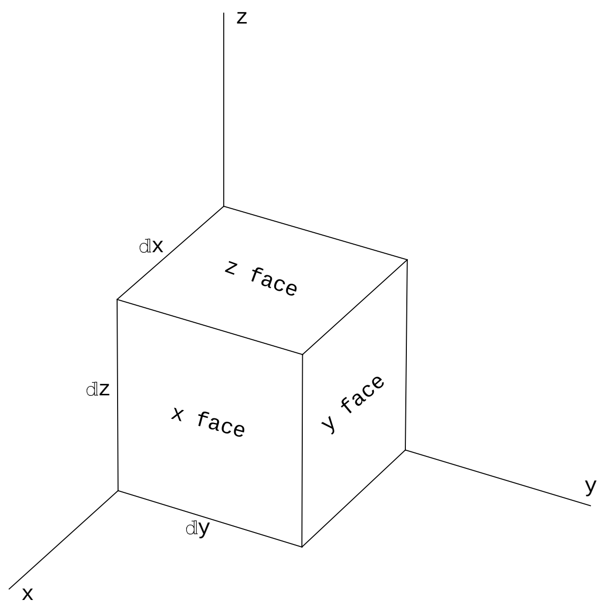

- Graphics -

The net amount of charge leaving the two z, faces is
$| \rho^+(x, y, z + dz) | v_z{}^+(x, y, z + dz) - | \rho^+(x, y, z) | v_z{}^+(x, y, z)$
$- | \rho^-(x, y, z + dz) | v_z{}^-(x, y, z + dz) + | \rho^-(x, y, z) | v_z{}^-(x, y, z)$
$= \left(\frac{\partial j_z}{\partial z}\, dz\right)(dx\, dy)\, dt$
$= \frac{\partial j_z}{\partial z}\, d^3 r\, dt.$

Similarly for the other four faces we get $\frac{\partial j_x}{\partial z}\, d^3 r\, dt$ and $\frac{\partial j_y}{\partial z}\, d^3 r\, dt$. Adding the three terms gives the total exodus and so charge conservation says that

$$\left(\frac{\partial j_x}{\partial z} + \frac{\partial j_y}{\partial z} + \frac{\partial j_z}{\partial z}\right) d^3 r\, dt = \vec{\nabla} \cdot \vec{j}\, d^3 r\, dt = -\frac{\partial}{\partial t}\,(\rho\, d^3 r)\, dt.$$

So we can express charge conservation as
$\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0.$

Notice that this expression is dimensionally correct; both terms have dimensions of charge/volume/time. We will use the full charge conservation relation later in the development of full electromagnetism, but for statics, we use a special case. In statics, we cannot have charge building up anywhere since that would violate the statics condition. Thus for magnetostatics, we have the very important condition on the current density:

$\vec{\nabla}\cdot\vec{j}=0$ (magnetostatics).

Summary of Experimental Results in Magnetostatics

Experimentally it was discovered that currents interact with one another according to a rather complicated rule. I take it as experimentally found that

$$\vec{F}_{\text{on 1 due to 2}} = \int_{\text{all space}} \left(\vec{j}_1(\vec{r})\times \vec{B}_{\text{due to 2}}(\vec{r})\right) d^3 r$$

where

$$\vec{B}_{\text{due to 2}}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\text{all space}} \frac{\vec{j}_2(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3 r\,'.$$

In contrast to the situation in electrostatics in which we started with point charges and then progressed to a distribution of charges, in magnetostatics, there is no mention of point current densities for which the above formulas are a generalization. The reason is that charge conservation prevents us from having a point current density *in statics*. In particular to get a point current density in the z direction at the origin, you could consider a current I flowing in a very short, very thin wire of length Δz with center at the origin. Then, take the appropriate limit. We will in fact consider just this situation in full-dress electromagnetism, but because of charge conservation, such a situation violates the *statics condition.* Putting balls on the ends of the wire to prevent accumulated charge there from producing an infinite potential, we get a charge on one ball of $-I\,t$ and on the other of $+I\,t$ assuming the current is constant and there is no charge on the balls at $t=0$.

So you see that we must know how to treat time-varying charges and time varying electric fields to handle the problem. All I what to show here is that there are no point current densities in magnetostatics.

SI Dimensions of magnetic and electrical things

The parameter μ_0 has numerical value $4\pi\,10^{-7}$ and appropriate units to make the equations dimensionally correct, *by definition.* To see what its dimensions must be, note that the first equation requires B to have dimensions of $\frac{\text{Newton second}}{\text{Coulomb meter}}$ and this is given the name of Tesla in honor of Nicola Tesla, who pushed the idea (considered nutty at the time) that alternating current was preferable to direct current in practical applications. Incidentally the name assigned conventionally to B is the *magnetic induction*, the name magnetic field being reserved for the H field, which in free space is simply $\vec{H}=\frac{\vec{B}}{\mu_0}$. Informally, most physicists call B magnetic field, and I'm sure I'll do that also.

Introducing the unit of current $I = \int_{\text{open surface}} \vec{j} \cdot d\vec{A}$ as the Ampere, we have

$$\text{Ampere} = \frac{\text{Coulomb}}{\text{second}}$$

So the dimensions of B are
$\text{Tesla} = \frac{\text{Newton}}{\text{Ampere meter}}$

We then get for the dimensions of μ_0, $\frac{\text{Tesla meter}}{\text{Ampere}}$. Later we will define the unit of inductance.

The dimension of inductance is
$\text{Henry} = \frac{\text{Tesla meter}^2}{\text{Ampere}}$ (after Joseph Henry of Princeton Univ.).

This gives us a more conveniently remembered way to express the units of μ_0: Henry/meter. So we get, by definition,

$$\mu_0 = 4\pi\, 10^{-7}\ \frac{\text{Henry}}{\text{meter}}.$$

As an addendum, we have for the dimensions of electrostatics things. From $F = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{r^2}$ we get the dimensions of ε_0 to be $\frac{\text{Coulomb}^2}{\text{Newton meter}^2}$. From $U = \frac{1}{2} Q\Phi$ we get that a (Coulomb Volt) is the same as a (Newton meter) so we get

$$\text{Volt} = \frac{\text{Newton meter}}{\text{Coulomb}}$$

Next, the dimensions of Capacitance are $\frac{\text{Coulomb}}{\text{Volt}}$ and this is called a Farad, for Michael Faraday.

$$\text{Farad} = \frac{\text{Coulomb}}{\text{Volt}} = \frac{\text{Coulomb}^2}{\text{Newton meter}}.$$

Comparing this the units worked out for the permittivity, ε_0, we find

$$\text{Dimensions of } \varepsilon_0 = \frac{\text{Farad}}{\text{meter}}$$

So in this form we see a nice symmetry between the units of μ_0 and ε_0. Finally, we can get the dimensions of the product $\mu_0\,\varepsilon_0$ as

$$\frac{\text{Farad Henry}}{\text{meter}^2}$$
$$= \frac{\text{Coulomb}^2}{\text{Newton meter}} \frac{\text{Tesla meter}^2 \text{ second}}{\text{Coulomb}} \frac{1}{\text{meter}^2}$$
$$= \frac{\text{Tesla Coulomb second}}{\text{Newton meter}}$$
$$= \frac{\text{Newton second}}{\text{Coulomb meter}} \frac{\text{Coulomb second}}{\text{Newton meter}}$$
$$= \frac{\text{seconds}^2}{\text{meter}^2}$$

and these are the dimensions of $\frac{1}{c^2}$ and in fact as we will show later, but already know, $\mu_0\,\varepsilon_0 = \frac{1}{c^2} \simeq \frac{1}{9{\cdot}10^{16}}$. Thus we get

$$\boxed{\varepsilon_0 \simeq \frac{1}{4\,\pi\,10^{-7}} \frac{1}{9\,10^{16}} = \frac{1}{4\,\pi} \frac{1}{9\,10^9} \frac{\text{Farad}}{\text{meter}}.}$$

Enough for dimensions!

■ The cross product by components

But before returning to magnetostatics, it is clear that an intimate familiarity with the cross product is going to be very important in dealing with the subject and so a few words about that subject first. Often it is much simpler to directly express the cross product in components in order to manipulate expressions than in its vector form. It is a useful exercise to learn how to flip back and forth between the two notations. To express the cross product by components compactly, the Levi-Civita symbol ϵ_{ijk} is used. This is the fully antisymmetric unit tensor of third rank (and is most useful in three dimensional space). Its definition is

$$\boxed{\epsilon_{ijk} = \begin{pmatrix} 0 & \text{if any two of its indices are the same} \\ 1 & \text{if its indices are 123, 231, or 312} \\ -1 & \text{if its indices are 132, 213, or 321} \end{pmatrix}.}$$

It then has the property that $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$(unchanged by any even permutation of the indices,i.e., a permutation that can be effected by an even number of interchanges of among its indices) and it changes sign under any odd permutation of its indices. It is then easy to verify that

$$\boxed{\left(\vec{a}\times\vec{b}\right)_i = \epsilon_{ijk}\,a_j\,b_k.}$$

From the familiar expression for a double cross product, namely,

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \vec{a}\cdot\vec{c}\,\vec{b} - \vec{a}\cdot\vec{b}\,\vec{c}$$

we get
$\epsilon_{ijk}\, a_j\, \epsilon_{kmn}\, b_m\, c_n = b_i\, a_j\, c_j - c_i\, a_j\, b_j = a_j\, b_m\, c_n (\delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm})$

Since the three vectors are arbitrary, we have the frequently used relation

$$\boxed{\epsilon_{ijk}\, \epsilon_{kmn} = \epsilon_{kij}\, \epsilon_{kmn} = \delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm}}$$

You can easily directly check this relation against the definitions.

It should also be noticed from the definition of the determinant of a three by three matrix, M_{ij}, that

$$\boxed{\det(M)\ \epsilon_{ijk} = M_{i\ell}\, M_{jm}\, M_{kn}\, \epsilon_{\ell mn}}$$

The relation between cross products and determinants is expressed by the well known expression

$$\vec{a} \times \vec{b} = \begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}.$$

Its time to get back to magnetostatics.

■ Back to Magnetostatics

Copying from above, I take it as experimentally found that

$$\vec{F}_{\text{on 1 due to 2}} = \int_{\text{all space}} \left(\vec{j}_1(\vec{r}) \times \vec{B}_{\text{due to 2}}(\vec{r}) \right) d^3 r$$
where
$$\vec{B}_{\text{due to 2}}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\vec{j}_2(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}\, d^3 r'.$$

The second of these formulas is called the Biot-Savart law.

The functional form in the position vectors is very familiar. So we use
$\frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right)$
to write
$$\vec{B}_{\text{due to 2}}(\vec{r}) = -\frac{\mu_0}{4\pi} \int_{\text{all space}} \vec{j}_2(\vec{r}\,') \times \vec{\nabla}\left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) d^3 r'$$
$$= \frac{\mu_0}{4\pi} \int_{\text{all space}} \vec{\nabla}\left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) \times \vec{j}_2(\vec{r}\,')\, d^3 r'.$$

Now notice that the differentiation is with respect to the primed coordinates and the integral is over the primed ones. So we can take the ∇ out from under the integral. But what about the cross product? To be sure all is well, express the equation in components (and drop the index "2". This yields

$$B_i(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \epsilon_{ijk} \nabla_j\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) j_k(\vec{r}\,')\, d^3 r'.$$

Since the ϵ_{ijk} is a constant we get

$$B_i(\vec{r}) = \epsilon_{ijk} \nabla_j\left(\frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{j_k(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3 r'\right)$$

So going back to vector notation, we get

$$\boxed{\begin{array}{l} \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) \qquad . \\ \text{where} \\ \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3 r' \end{array}}$$

So we have the usual formula for the electrostatic scalar potential in triplicate, and so the appropriate name for $\vec{A}$ is the vector potential. Furthermore, because the divergence of a curl is identically zero, we have the important relation

$\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0.$

And next from $\nabla^2\left(\frac{1}{\vec{r}-\vec{r}\,'|}\right) = -4\pi\,\delta^3(\vec{r} - \vec{r}\,')$ we get in Cartesian coordinates

$\nabla^2 A_i = -\mu_0\, j_i.$

Also by taking the curl of the magnetic field, and expanding the double cross product, we get

$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A}\,.$

We have the Laplacian of the above vector potential, but what is its divergence? We can move the derivatives to inside the integral since they are with respect to the unprimed coordinates. So we get

$$\vec{\nabla} \cdot \vec{A} = \frac{\mu_0}{4\pi} \int_{\text{all space}} \vec{\nabla} \cdot \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3 r'$$
$$= \frac{\mu_0}{4\pi} \int_{\text{all space}} \vec{j}(\vec{r}\,') \cdot \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3 r'$$

Now as usual, exploit $\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$ to get

$$\vec{\nabla} \cdot \vec{A} = -\frac{\mu_0}{4\pi} \int_{\text{all space}} \vec{j}(\vec{r}\,') \cdot \vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3 r'.$$

As usual, do a partial integration to get

$$\vec{\nabla} \cdot \vec{A} = -\frac{\mu_0}{4\pi} \int_{\text{surface at }\infty} \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) \vec{j}(\vec{r}\,') \cdot d A' + \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\vec{\nabla}' \cdot \vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3 r'$$

Withe assumption that the current density falls to zero faster than $\frac{i}{r}$, the surface integral vanishes. Because of the statics version of charge conservation, the second integral also vanishes. So we get for the above vector potential that $\vec{\nabla} \cdot \vec{A} = 0$. Putting it all together we have the fundamental equation in $\vec{B}$ for magnetostatics:

$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}.$$

Clearly the first equation implies there is a vector potential satisfying $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$ and the second equation provides the statics version of charge conservation, $\vec{\nabla} \cdot \vec{j} = 0$. Just as in electrostatics the vector potential is not unique; what we have above is a sufficient vector potential for obtaining the magnetic field, but we could add to it the gradient, $\vec{\nabla}\psi$,of any nice, smooth function, ψ, and get the same magnetic field. The reason, of course, is that the curl of such a gradient identically vanishes. We see that there is a far richer range of possibilities of vector potentials for a given B field than the corresponding case in electrostatics. There we could only add a constant to a potential to get an equivalent one (in two dimensional electrostatics, there is more freedom). In general these are called gauge transformations, and electromagnetism is said to symmetric under these gauge transformations.

■ Two simple idealized examples

■ Long straight wire with symmetric return circuit

First consider a very long wire carrying a current I with its "return" circuit symmetrically disposed about the axis of the long wire so the problem has circular symmetry about this axis. Note that we need a return circuit carrying total current I in the opposite direction (so the name "return") to conserve charge!

Consider a circular area A of radius ρ centered on the wire with normal parallel to the wire. I assume that the symmetric return circuit is outside of this area so the only current e3nsity that it includes is that in the wire. Then, by symmetry, the component of the B field in the direction of the wire *due to the current in the wire and its return circuit* must vanish. This follows directly from the Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\vec{j}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \, d^3 r'$$

If all of $\vec{j} = \vec{e}\, j$ where $\vec{e}$ is a constant unit vector, then clearly $\vec{e} \cdot B = 0$. So we only need consider components of $\vec{B}$ in the plane of A and, by translational symmetry along the wire, these are independent of distance along it. By rotational symmetry about the axis of the wire, at every point on the circumference of the circle bounding A, $\vec{B}$ have the same magnitude and must make the same angle with respect

to the radial vector from the axis to the point. But from
$\int_V \vec{\nabla} \cdot \vec{B}\, d^3 r = \int_{\partial V} \vec{B} \cdot d\vec{A} = 0$
applied to a cylinder of unit length along the wire whose ends are two versions of the area A, we see that the radial component of $\vec{B}$ must vanish. Finally, we get the familiar formula for the circumferential component of B is obtained from $\int_A \vec{\nabla} \times \vec{B} \cdot d\vec{A} = \mu_0 \int_A \vec{j} \cdot d\vec{A}$.
Use Stoke's theorem on the left hand side and the definition of the current I on the right side to get
$2\pi\rho B_\varphi = \mu_0 I$ or $B_\varphi = \frac{\mu_0 I}{2\pi\rho}$.
It is important to notice that this applies just inside the return circuit which entered the problem only by its assumed circular symmetry!

To get an idea of size, if $I = 1$ Amp, $\rho = 1$ cm, we get a magnetic field of $2\,10^{-5}$ Tesla. Many physicists prefer to use the cgs unit for magnetic field, the gauss. There are 10^4 Gauss in 1 Tesla, so the above field is 0.2 Gauss. The Earth's magnetic field is of order 1/2 Gauss where we walk around, the same order of size.

■ Long, straight uniform solenoid

To get a bigger field, concentrate the current. Ampere originally did this by making a flat tight spiral of wire, and it is possible to calculate the axial component of field in this case. But a nicer arrangement for experiments and also calculation is the idealized solenoid. I will assume a tight spiral winding of N turns per unit length along the length of the solenoid, carrying a current I. Approximate the current by ignoring its longitudinal part, i.e., assume it has only a circumferential component. And assume the cross section of the solenoid is a circle.

Then by symmetry that is no component of the field in the plane whose normal is along the axis of the solenoid - there is only a B field parallel to its axis. I know of no symmetry argument that gives the result that, for a long solenoid, the magnetic field outside the solenoid (of any cross sectional shape) is negligible compared to that inside it, I will just assume it. We will get this important result later It is then easy to use Stoke's theorem to get the field anywhere inside the solenoid. Set up a closed loop with one leg of unit length inside the solenoid and parallel to the axis, and close it with two segments normal to the axis and passing from inside to outside the solenoid. The segment outside the solenoid can be anything joining the two ends sticking out. Applying Stoke's theorem to the are defined by this loop immediately yields the result that
$B_{\text{axis of solenoid}} = \mu_0 N I$
Note that his has the proper dimensions since N has dimensions of reciprocal length. As a numerical example, suppose you have wire od diameter 1 mm including its insulation. Then a tight winding has 1000 turns per meter, and if it carries 1 Amp, the field is
$4\pi\, 10^{-7}\, 10^3$ Tesla $\simeq 1.3\, 10^{-3}$ Tesla $= 13$ Gauss.

▪ Magnetic fields in current free regions of space

It is very interesting to notice that in current-free regions of space, the laws of magnetostatics are exactly the same as those for electrostatics in charge-free regions of space.

In current-free and charge-free regions

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\cdot\vec{E} = 0$$
$$\vec{\nabla}\times\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = 0.$$

▪ Magnetic Multipoles

Therefore what we know about electrostatics in these circumstance is immediately applicable to magnetostatics, in particular, multipole expansions. The relation of the moments to the currents generating the fields, is of course different form what we have in electrostatics, but the geometric structure of the fields are just the same. From our earlier expression for the vector potential, we can immediately write a multipole expansion for the vector potential for the case that we are interested in distant fields from a localized current distribution. Then we have

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\text{all space}} \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3 r'$$
$$= \frac{\mu_0}{4\pi}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\,\frac{1}{r^{\ell+1}}\,\vec{m}^{\text{sh}}_{\ell m}\, Y_{\ell m}(\theta,\varphi)$$

where

$$\vec{m}^{\text{sh}}_{\ell m} = \int \vec{j}(\vec{r})\, r^{\ell}\, Y^{*}_{\ell m}(\theta,\varphi)\, d^3 r$$

are the vector multipole moments developed from the spherical harmonics - that what the "sh" superscript is intended to remind you of.
Similarly, if you are interested in the fields near the origin and all of the current are far away, you get the "inside" form of the multipole expansion

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\, r^{\ell}\, \vec{M}^{\text{sh}}_{\ell m}\, Y_{\ell m}(\theta,\varphi)$$

where

$$\vec{M}^{\text{sh}}_{\ell m} = \int \vec{j}(\vec{r})\, \frac{1}{r^{\ell+1}}\, Y^{*}_{\ell m}(\theta,\varphi)\, d^3 r.$$

If all of the current in a problem is on the surface of a sphere and is suitably distributed, you can excite a pure multipole field of arbitrary order outside the sphere and the corresponding multipole field inside the sphere. This is qualitatively the same as we have in electrostatics, but there are important technical differences that arise from the charge conservation condition and the occurrence of cross products rather than gradients to relate the potentials to the magnetic and electric fields respectively. For example, there is no monopole field in the magnetic case, and the inside magnetic field of a dipole is oppositely directed (for the same geometry of the outside field) compared to an inside electric field.

I will only discuss in detail the monopole and dipole cases, and these only for the outside fields..

- **Magnetic monopoles**

First the monopole case. For the outside field, we only have to consider

$$\vec{m}_{00}^{\text{sh}} = \frac{1}{\sqrt{4\pi}} \int \vec{j}(\vec{r})\, d^3 r$$

If you think of the current as being conducted in some closed loops of wire all bunched up near the origin, then it is easy to see that this integral vanishes. This just follows from the relation for a thin wire carrying a current I. In this case we have the important relation

$$\int_{\Delta \text{V}} \vec{j}(\vec{r})\, d^3 r = \int_{\Delta \ell} \int_{\delta \text{A}} \vec{e}\, \frac{I}{\Delta \text{A}}\, dA\, d\ell = I\, \vec{e}\, \Delta \ell$$

where ΔV is a volume of length $\Delta\,\ell$ along the wire and capped by differential areas ΔA at the two ends. The unit vector $\vec{e}$ is tangent to the wire at the point in question. If we integrate over the whole loop of a wire we have

$$\int \vec{j}(\vec{r})\, d^3 r = I \oint_{\text{loop}} \vec{e}(\vec{r})\, d\ell$$

and for a closed loop this is just zero and so there is no monopole moment. A more formal way to get this result as a consequence of $\vec{\nabla} \cdot \vec{j} = 0$ is to notice that

$$\vec{\nabla} \cdot (r_i\, \vec{j}) = \nabla_j (r_i\, j_j) = \delta_{ij}\, j_j + r_i\, (\nabla_j\, j_j) = j_i + r_i \vec{\nabla} \cdot \vec{j}$$

Charge conservation makes the divergence term vanish and so we get for a volume V
big enough to contain all of the currents so that $\vec{j} = 0$ outside of V and *on its surface*. So we get by integrating the above relation that

$$\boxed{\int_V \vec{\nabla} \cdot (r_i\, \vec{j})\, d^3 r = \int_V j_i\, d^3 r = \int_{\partial V} r_i\, \vec{j} \cdot d\vec{A} = 0.}$$

So we get the important result that for outside fields the all three components of the monopole moment vector vanish. Incidentally, for the inside fields, we needn't bother with evaluating $\vec{M}_{00}^{\text{sh}}$ since it just contributes a constant to the vector potential and so can be "gauged away", i.e., you can make a gauge transformation that eliminates it. Of course, a constant is a trivial case of using gauge symmetry.

■ Magnetic Dipole, outside field

So the lowest order term is both series is the dipole term. As we know from electrostatics, we needn't bother with the spherical harmonics and all of their irritating normalization factors. It is easy to get for the outside field that

$$\vec{A}^{\text{dipole}}_{\text{outside}} = \frac{\mu_0}{4\pi}\left(\vec{m}^{\text{sh}}_x\, x + \vec{m}^{\text{sh}}_y\, y + \vec{m}^{\text{sh}}_z\, z\right)\frac{1}{r^3} = \frac{\mu_0}{4\pi}\,\vec{m}^{\text{sh}}_i\,\frac{r_i}{r^3}$$

where

$$\vec{m}^{\text{sh}}_i = \int r_i\,\vec{j}(\vec{r})\,d^3 r$$

By the way, I continue to include the irritating "sh" superscript so as to reserve the unadorned "m" name for the vectorial magnetic dipole moments that we will get to shortly.

If we write this in components, we get

$$(\vec{m}^{\text{sh}}_i)_j = \int r_i\, j_j(\vec{r})\,d^3 r$$

and it is natural to write $(\vec{m}^{\text{sh}}_i)_j = m^{\text{sh}}_{ij}$ where the second index indicates to which vector component of the vector potential the contributes to. So we see that for the dipole multipole we get a second rank tensor to deal with, a beast with nine components.

Let'ssee what are the consequences of charge conservation on the dipole tensor. Consider

$$\vec{\nabla}\cdot(r_i\, r_j\,\vec{j}) = \nabla_k(r_i\, r_j\, j_k) = \delta_{ik}\, r_j\, j_k + r_i\,\delta_{jk}\, j_k + r_i\, r_j(\nabla_k\, j_k) = r_j\, j_i + r_i\, j_j.$$

where I have used charge conservation. So we get

$$r_j\, j_i = -r_i\, j_j$$

which implies the interesting result that, of the nine components of the matrix, only three are independent. In particular

$$m^{\text{sh}}_{ij} = \int r_i\, j_j\, d^3 r = -\int r_j\, j_i\, d^3 r = -m^{\text{sh}}_{ji}$$

Therefore, $m^{\text{sh}}_{11} = m^{\text{sh}}_{22} = m^{\text{sh}}_{33} = 0$, and $m^{\text{sh}}_{21} = -m^{\text{sh}}_{12}$, $m^{\text{sh}}_{31} = -m^{\text{sh}}_{13}$, $m^{\text{sh}}_{32} = -m^{\text{sh}}_{23}$.

This can be more elegantly expressed using ϵ_{ijk}. In particular, define the three moments m_k by

$$m^{\text{sh}}_{ij} = \epsilon_{ijk}\, m_k.$$

This explicitly expresses the antisymmetry of the $m^{\rm sh}$ matrix, and it is easy to invert Contract both sides with $\epsilon_{i j \ell}$ to get

$$\epsilon_{i j \ell}\, m_{i j}^{\rm sh} = \epsilon_{i j \ell}\, \epsilon_{i jk}\, m_k = (\delta_{j j}\, \delta_{\ell k} - \delta_{jk}\, \delta_{\ell j})\, m_k = 3\, m_\ell - m_\ell = 2\, m_\ell$$

So we get the nice formula for the three independent components of the magnetic dipole moment, $\vec{m}$.

$$m_\ell = \frac{1}{2}\, \epsilon_{i j \ell}\, m_{i j}^{\rm sh} = \frac{1}{2}\, \epsilon_{i j \ell} \int r_i\, j_j(\vec{r})\, d^3 r = \frac{1}{2} \int (\vec{r} \times \vec{j}\,)_\ell\, d^3 r$$

or dipole field is fully defined by the vector

$$\boxed{\vec{m} = \frac{1}{2} \int \vec{r} \times \vec{j}\, d^3 r.}$$

It is interesting and useful to express this magnetic dipole moment when the current is carried in a thin wire. In this case as discussed above we take the delta functions in the current density in to account by the formal substitution of

$$\boxed{\int_V \ldots\, \vec{j}\, d^3 r \to I \oint_{\text{wire circuit}} \ldots\, d\vec{\ell}.}$$

So for a thin wire circuit carrying a current I we get
$$\vec{m} = I\, \frac{1}{2} \oint_{\text{circuit}} \vec{r} \times d\vec{\ell} = I \int_{\text{area of circuit}} d\vec{A}$$

since $\frac{1}{2}\, \vec{r} \times d\vec{\ell}$ is the area of a triangle defined by $\vec{r}$, $d\vec{\ell}$, and $\vec{r} + d\vec{\ell}$ and each of these forms a differential element if the area of the circuit. It may be hard to visualize when the circuit is all bunched up like a wad of paper, but such a circuit does have an area!

But we still need to express the vector potential in terms of this new vector. From above we have

$$\vec{A}_{\text{outside}}^{\text{dipole}} = \frac{\mu_0}{4\pi}\, \vec{m}_i^{\rm sh}\, r_i\, \frac{1}{r^3}$$

or

$$r^3 \left(A_{\text{outside}}^{\text{dipole}}\right)_j = \frac{\mu_0}{4\pi}\, (\vec{m}_i^{\rm sh}\,)_j\, r_i = \frac{\mu_0}{4\pi}\, m_{i j}^{\rm sh}\, r_i = \frac{\mu_0}{4\pi}\, \epsilon_{i jk}\, m_k\, r_i = \frac{\mu_0}{4\pi}\, \epsilon_{jki}\, m_k\, r_i = \frac{\mu_0}{4\pi}\, (\vec{m} \times \vec{r})_j$$

So we get the important result

$$\vec{A}^{\text{dipole}}_{\text{outside}} = \frac{\mu_0}{4\pi} \frac{\vec{m}\times\vec{r}}{r^3}$$

$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{j}\, d^3 r.$$

Checking dimensions is always a useful business. From the general relation for the vector potential we see that $\frac{\vec{A}}{\mu_0}$ has the dimensions of current. The dipole moment has dimensions of (current · area), so the dimensions check in the dipole field expression.

■ Magnetic Dipole, inside field

As above we can write

$$\vec{A}^{\text{dipole}}_{\text{inside}} = \frac{\mu_0}{4\pi}\left(\vec{M}^{\text{sh}}_x\, x + \vec{M}^{\text{sh}}_y\, y + \vec{m}^{\text{sh}}_z\, z\right) = \frac{\mu_0}{4\pi}\,\vec{M}^{\text{sh}}_i\, r_i$$

where similarly to above we have the matrix

$$M^{\text{sh}}_{ik} = \int \frac{r_i\, j_k}{r^3}\, d^3 r$$

with $\left(A^{\text{dipole}}_{\text{inside}}\right)_k = \frac{\mu_0}{4\pi}\, M^{\text{sh}}_{ik}\, r_i$

However, this matrix need not be antisymmetric, as in the outside field case. As an example consider a loop of thin wire making a circular arc of radius b from $\varphi = \alpha$ to $2\pi - \alpha$ and a circular arc of radius a from $\varphi = -\alpha$ to α. The arcs are joined at their ends by radial bits of wire. You should get for this case that $M^{\text{sh}}_{21} \neq M^{\text{sh}}_{12}$. However, by making a gauge transformation we can calculate the fields with modified inside dipole moments. Let $a_{ij} = \frac{1}{2}\left(M^{\text{sh}}_{ik} + M^{\text{sh}}_{ki}\right)$, which is symmetric and, in fact, just the symmetric part of M^{sh}_{ik}. Then choose the gauge field $\psi = \frac{\mu_0}{4\pi}\frac{1}{2}\, a_{ij}\, r_i\, r_j$ so that $\left(A^{\text{dipole}}_{\text{inside}}\right)_k$ gets replaced by itself minus $\frac{\mu_0}{4\pi}\left(\frac{1}{2}\, a_{ij}\,\delta_{ik}\, r_j + \frac{1}{2}\, a_{ij}\, r_i\,\delta_{jk}\right) = \frac{\mu_0}{4\pi}\left(\frac{1}{2}\, a_{kj}\, r_j + \frac{1}{2}\, a_{ik}\, r_i\right)$. Use the symmetry of the a matrix and change the names of summing indices as necessary to write this as $\frac{\mu_0}{4\pi}\, a_{ik}\, r_i$. So we get the modified vector potential (I'll continue to call it by the same name - it is already too heavy with scripts.)

$$\left(A^{\text{dipole}}_{\text{inside}}\right)_k = \frac{\mu_0}{4\pi}\left(M^{\text{sh}}_{ik} - a_{ik}\right) r_i = \frac{\mu_0}{4\pi}\left(\frac{1}{2}\left(M^{\text{sh}}_{ik} - M^{\text{sh}}_{ki}\right)\right) = \frac{\mu_0}{4\pi}\, M^{\text{sh}\,\prime}_{ik}\, r_i.$$

The new dipole moments for the inside fields are manifestly antisymmetric

$$M^{\text{sh}\,\prime}_{ik} = \frac{1}{2}\int \frac{r_i\, j_k - r_k\, j_i}{r^3}\, d^3 r$$

and so we can construct the magnet dipole moments for inside fields. The results are

$$M_\ell = \frac{1}{2}\,\epsilon_{ik\ell}\, M^{\text{sh}\,\prime}_{ik} = \frac{1}{4}\,\epsilon_{\ell ik}\int \frac{r_i\, j_k - r_k\, j_i}{r^3}\, d^3 r = \frac{1}{2}\int \frac{\left(\vec{r}\times\vec{j}\right)_\ell}{r^3}\, d^3 r$$

and

$$\vec{A}_{\text{inside}}^{\text{dipole}} = \frac{\mu_0}{4\pi} \vec{M} \times \vec{r}$$

$$\vec{M} = \frac{1}{2} \int \frac{\vec{r} \times \vec{j}}{r^3} \, d^3 r$$